# PH 350 Problem Set # 5

Due: Tuesday, April 4, 2006

1. Thermally excited string. In this problem we consider an ideal string, which is made of a material that is insensitive to thermal effects. When we heat this string, we will excite its normal modes in a thermal distribution. Because of this assumption, this model is not a particularly good description of real strings; however, we will see that it does a good job of describing the excitation of *fields*; roughly, we can think of the displacement of the string as corresponding to the amplitude of the electric field for black-body radiation in one dimension. We will also restrict the string to only vibrate in one plane — which is analogous to considering only one polarization of the electromagnetic wave (and including both independent planes of oscillation will just introduce the same factor of 2 in the final results).

   We consider a string of length $L$, whose displacement at position $x$ and time $t$ is given by $\phi(x, t)$. We assume that the ends are nailed down, so $\phi(x = 0, t) = \phi(x = L, t) = 0$. It energy is then

   $$U = \int_0^L \left( \frac{1}{2}\lambda\dot{\phi}^2(x,t) + \frac{1}{2}\theta\phi'(x,t)^2 \right) dx$$

   where $\dot{\phi} = \dfrac{\partial\phi}{\partial t}$ and $\phi' = \dfrac{\partial\phi}{\partial x}$. Here $\lambda$ is the mass density of the string and $\theta$ is its tension. So the first term gives the kinetic energy due to the motion of the string, and the second term gives the potential energy in stretching it.

   As you know from PH212, the modes of oscillation of this string, subject to the boundary conditions, are given by

   $$\phi_j(x,t) = A_j \sin(k_j x)\cos(\omega t) + B_j \sin(k_j x)\sin(\omega t)$$

   for $j \in 1, 2, 3, \ldots$, where the wave number is $k_j = \dfrac{\pi j}{L}$, the frequency is $\omega_j = v k_j$, and the speed of wave propagation is $v = \sqrt{\dfrac{\theta}{\lambda}}$.

   (a) Consider a particular mode $j$. Find the energy in this mode for given amplitudes $A_j$ and $B_j$.

   (b) Compute the average classical energy found in this mode at temperature $T$. (You'll need to integrate over all values of both $A_j$ and $B_j$ in computing the partition function.)

   (c) Sum the previous result over all values of $j$ to obtain the average total energy. You should find a problem. What happened?

   (d) In the previous calculation, we computed the total energy by summing the square of $\dot{\phi}$ and $\phi'$ over all modes. But really we should have taken the square of the sum instead. Explain why this manipulation is justified, and is therefore *not* the cause of the problem you ran into in the previous part. (This is really a PH212 problem; if you are stuck try computing the energy for a superposition of just two modes.)

   Now let's fix the problem in part (c) by analogy with blackbody radiation. What we learned in that case was that we cannot choose the amplitude of each mode arbitrarily. Instead we should *quantize* the excitations of each mode $j$, so that the energy stored in mode $j$ is $n_j\hbar\omega$, for $n_j = 0, 1, 2, 3\ldots$. (We are ignoring any contributions from zero-point energies.)

(e) Find an expression for the normalized probability $p_j(n_j)$ that mode $j$ has exactly $n_j$ excitations. Find the average number of excitations $\bar{n}_j$ in each mode.

(f) Find the average energy in each mode and the average total energy at temperature $T$ using the quantum approach. For the total energy, you should approximate the sum over modes by an integral in the same way as we did for blackbody radiation. Also find the standard deviation of the average energy, which you computed on an earlier problem set is equal to $T\sqrt{k_B C_V}$ where $C_V = dU/dT$.

(g) For use below, find an expression for the total amplitude $C_j = \sqrt{A_j^2 + B_j^2}$ in a particular mode given the occupation number $n_j$ by finding $C_j$ such that the classical energy matches up with the quantum result.

Now we would like to generate some real data to see how this system works. For this you will need to use Mathematica, Matlab, Maple, or an equivalent package. Note that I've provided skeleton code for Mathematica (also available in electronic form on the server) which you are welcome to use to help you set up the computer calculation.

(h) The first thing we need to be able to do is to randomly choose $n_j$ for each mode $j$ using the probability distribution $p_j(n)$ you computed above. Most mathematical packages can generate a random number between 0 and 1 (in Mathematica, for example, the `Random[]` function does this). How do we turn that result into a value for $n_j$? Well, suppose that the random number is $r$. Then we could choose $n_j$ to be largest integer such that

$$r \leq \sum_{n=n_j}^{\infty} p_j(n)$$

Then as $r$ varies from 0 to 1, $n_j$ goes from $\infty$ to 0 with the correct weighting. Analytically evaluate the right-hand side of this expression and solve for $n_j$ as a function of $r$. Remember that $n_j$ must be an integer; you may find it helpful to use the "floor" function $\lfloor x \rfloor$, which is the greatest integer less than or equal to $x$.

(i) We now can implement the calculation numerically. For simplicity, work in units where $\hbar = 1$, $\lambda = 1$, $\theta = 1$ and $k_B = 1$ (which corresponds to choosing convenient units of mass, length, time, and temperature). Then the only parameters to specify are the temperature $T$ and length of the string $L$. For the remainder of the problem, fix a large but reasonable value for $L$ (around 50 should be fine). Then, given $T$, you can randomly determine the number of quanta in each mode using the algorithm from the previous part, and from this find $C_j = \sqrt{A_j^2 + B_j^2}$. Pick a random phase $\chi_j$ between 0 and $2\pi$, and then let $A_j = C_j \cos\chi_j$ and $B_j = C_j \sin\chi_j$. Then you can sum over modes to obtain $\phi$.

Generate plots of a sample configuration $\phi(x, t = 0)$ for $T = 1$, $T = 5$, and $T = 10$. Note that on the computer you can only sum a finite number of $j$ values, so you should truncate the sum over modes at an appropriately high value. Why is it now safe to do this in the quantum case, when large $j$ gave a problem in the classical case?

(j) For each of the three temperatures above, compute the total energy of the system several times. (Your results will vary, since you're choosing different random numbers each time.) Compute the mean and the standard deviation of the total energies you found, and compare these results with the predicted values you calculated above.