Quantum Mechanics is Linear Algebra

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Linear Algebra Cheat Sheet

Column vector (quantum state): $|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix}$

Row vector (dual state): $\langle w| = (\, w_1^* \quad w_2^* \quad \ldots\,)$

Inner product: $\langle w|v\rangle = w_1^* v_1 + w_2^* v_2 + \ldots$

Linear operator: $\hat{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ \vdots & & \ddots \end{pmatrix}$

Eigenvector & eigenvalue: $\hat{A}|\lambda\rangle = \lambda|\lambda\rangle$

Average (expectation value): $\langle A\rangle_\psi = \langle\psi|\hat{A}|\psi\rangle$

Standard deviation (uncertainty): $\Delta A_\psi = \sqrt{\langle\left(\hat{A} - \langle A\rangle_\psi\right)^2\rangle_\psi}$

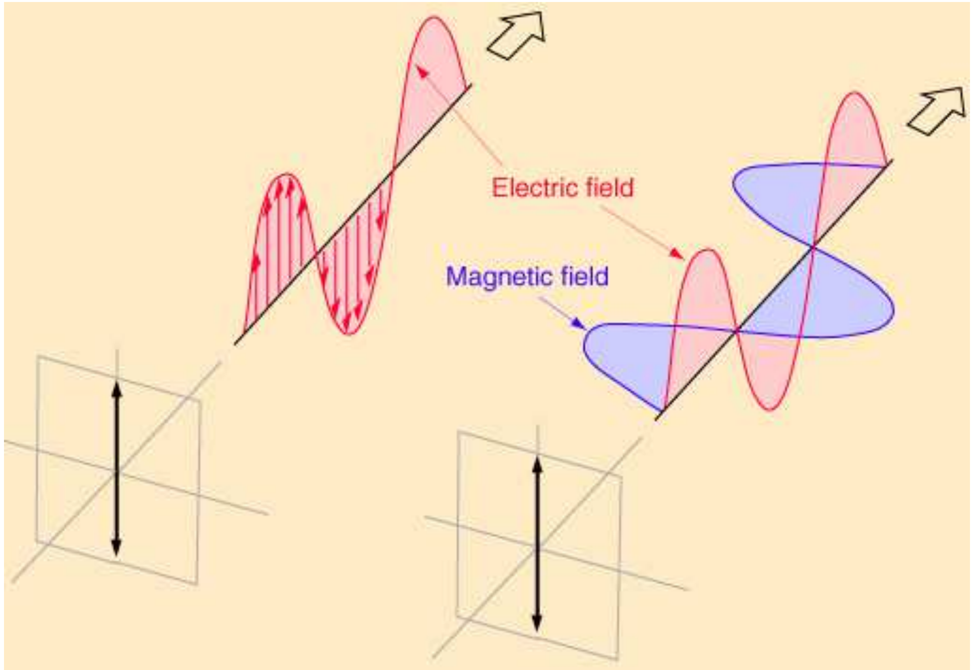
operator	eigenstates	eigenvalues
$\hat{O}_{\updownarrow} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\lvert \updownarrow \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\lvert \leftrightarrow \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	1 0
$\hat{O}_{\leftrightarrow} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\lvert \leftrightarrow \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\lvert \updownarrow \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	1 0
$\hat{O}_{\neswarrow} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\lvert \neswarrow \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lvert \nwsearrow \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	1 0
$\hat{O}_{\nwsearrow} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$\lvert \nwsearrow \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\lvert \neswarrow \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	1 0
$\hat{O}_{\theta} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$ [the general case]	$\lvert 1 \rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $\lvert 0 \rangle = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$	1 0

The Physics

Electromagnetic waves (radio waves, microwaves, visible light, X-rays, gamma rays, etc.) have a polarization: a vector that lives in the two-dimensional vectorspace perpendicular to the direction of the wave's propagation. It gives the axis along which the wave's electric field oscillates.

We'll hold fixed the frequency, wavelength, and direction of propagation of the wave, so that we just focus on polarization.

[Image from Georgia State University HyperPhysics]



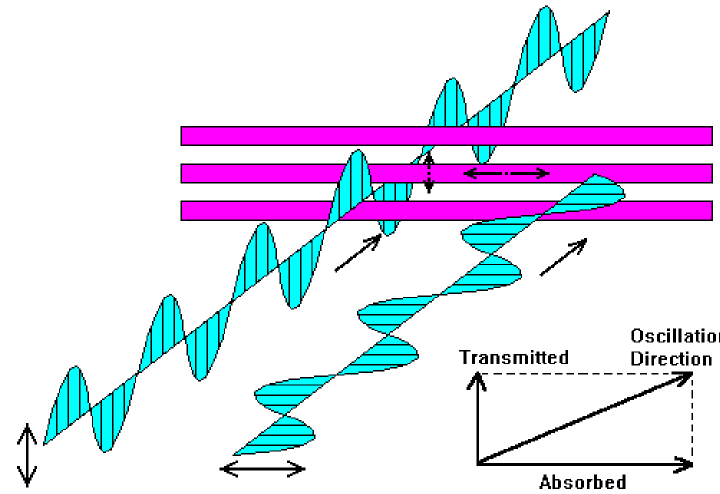
Working with polarization

A polarizing filter only allows light polarized along one axis to pass through. More precisely, it projects the polarization onto this axis.

Parallel: all gets through

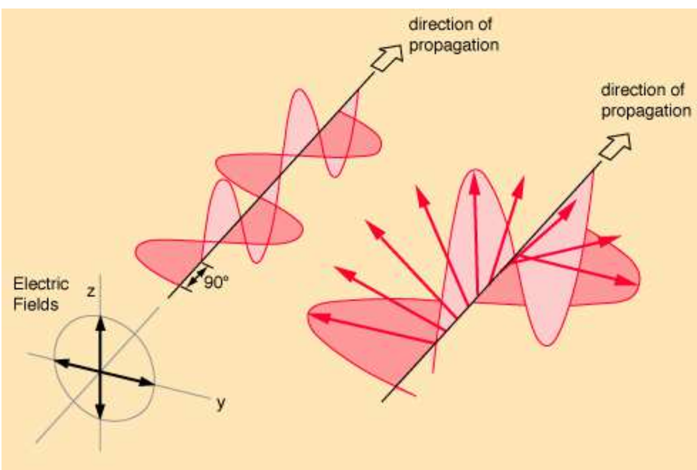
Perpendicular: none gets through

(In between: some gets through)



[Image from Steven Dutch, Univ. of Wisconsin, Green Bay]

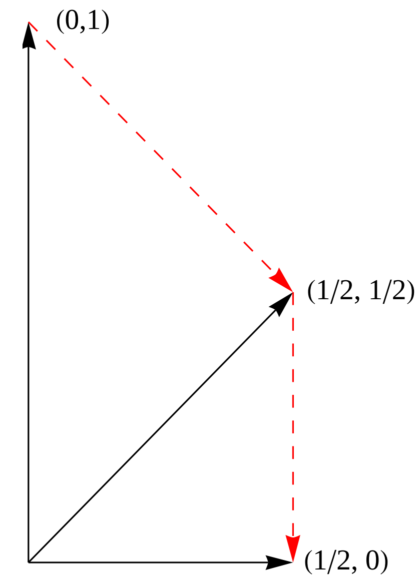
Note: filter only reduces the amount of light getting through.



Aside: the polarization vector can also be complex. Then the two components are out of phase and we get circular polarization.

[Image from Georgia State University HyperPhysics]

A curious example



If vertically polarized light is incident on a horizontal polarizer, nothing gets through.

But if vertically polarized light is incident on a 45° polarizer and then a horizontal polarizer, some does get through.

Even though a polarizer only blocks light, it can cause more light to get through!

Note: The intensity of light (how much energy it carries) is proportional to the square of the length of these vectors.

The problem: In quantum mechanics, light comes in discrete units, known as photons. Each photon has to make its own decision: Do I get through or not?

Postulate #1:

The state of a quantum system is completely described by a normalized vector $|\psi\rangle$ in a (complex) vectorspace.

- The vectorspace may be finite or infinite dimensional.
- There are no "hidden variables" — there is no other information available about the system besides what we can find out from this vector.

Example

Vertical polarization: $|\updownarrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Horizontal polarization: $|\leftrightarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Diagonal polarization: $|\swarrow\!\!\!\nearrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

or $|\nwarrow\!\!\!\searrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Postulate #2:

Any physically measurable quantity is represented by a Hermitian linear operator $\hat{\mathcal{O}} = \hat{\mathcal{O}}^\dagger$.

- A Hermitian operator is one whose conjugate equals its transpose.
- Eigenvalues of a Hermitian operator are always real.
- Eigenvectors of a Hermitian operator form an orthonormal basis.

Example

"Is polarization vertical?": $\hat{O}_{\updownarrow} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

"Is polarization horizontal?": $\hat{O}_{\leftrightarrow} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

"Is polarization diagonal?": $\hat{O}_{\nearrow\!\!\!\swarrow} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

or $\hat{O}_{\nwarrow\!\!\!\searrow} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Postulate #3:

The possible results of a measurement of a physical quantity are the eigenvalues λ_i of the associated operator $\hat{\mathcal{O}}$.

- Since $\hat{\mathcal{O}}$ is Hermitian, the eigenvalues are real.
- Physics is quantized!
- Many seemingly continuous physical quantities (energy, angular momentum, etc.) are actually the result of averaging discrete quantum outcomes over many particles.

Example

$\hat{O}_{\updownarrow}$, $\hat{O}_{\leftrightarrow}$, $\hat{O}_{\swarrow\!\!\!\nearrow}$, and $\hat{O}_{\nwarrow\!\!\!\searrow}$ all have the same eigenvalues:

- 1 = "yes" (photon gets through)
- 0 = "no" (photon blocked)

Note: No "maybe"!

Postulate #4:

The probability of measuring the eigenvalue λ_i is $P_i = |\langle\lambda_i|\psi\rangle|^2$, where $|\lambda_i\rangle$ is the normalized eigenvector of $\hat{\mathcal{O}}$ associated with the eigenvalue λ_i.

- $\langle\lambda_i|\psi\rangle$ is the i^{th} coordinate of $|\psi\rangle$ in the basis $\{|\lambda_i\rangle\}$.
- Probabilities sum to the norm squared of $|\psi\rangle$, which is 1.
- If our state $|\psi\rangle$ is itself an eigenstate $|\lambda\rangle$ of $\hat{\mathcal{O}}$, then we know for certain that the result of the measurement will be λ.

  If $|\psi\rangle$ is not an eigenstate, there are multiple possible outcomes.

Example

- Start in state $|\updownarrow\rangle$ and measure $\hat{O}_{\updownarrow}$:
  Probability of "yes" is $P_{\updownarrow} = |\langle\updownarrow|\updownarrow\rangle|^2 = 1$ and
  probability of "no" is $P_{\leftrightarrow} = |\langle\leftrightarrow|\updownarrow\rangle|^2 = 0$.
- Start in state $|\updownarrow\rangle$ and measure $\hat{O}_{\swarrow\!\!\!\nearrow}$:
  Probability of "yes" is $P_{\swarrow\!\!\!\nearrow} = |\langle\swarrow\!\!\!\nearrow|\updownarrow\rangle|^2 = \frac{1}{2}$ and
  probability of "no" is $P_{\nwarrow\!\!\!\searrow} = |\langle\nwarrow\!\!\!\searrow|\updownarrow\rangle|^2 = \frac{1}{2}$.

Postulate #5:

If we measure $\hat{\mathcal{O}}$ and find λ_i, then immediately after that measurement, the state suddenly jumps into the state $|\lambda_i\rangle$.

- Measurement changes the system!
- Two measurements in a row of the same quantity yield the same result.

Example

- If we start in state $|\updownarrow\rangle$ and measure $\hat{O}_{\leftrightarrow}$, we are certain to get the answer "no": $P_{\leftrightarrow} = 0$.
- But if we first measure $\hat{O}_{\swarrow\!\!\nearrow}$, there is a 50% probability we get the answer "yes." If so, we land in the state $|\swarrow\!\!\nearrow\rangle$. Now a measurement of $\hat{O}_{\leftrightarrow}$ has a 50% chance to yield the answer "yes" again. So

$$P_{\swarrow\!\!\nearrow,\leftrightarrow} = |\langle \leftrightarrow | \swarrow\!\!\nearrow \rangle|^2 \cdot |\langle \swarrow\!\!\nearrow | \updownarrow \rangle|^2 = \frac{1}{4} \neq 0$$

and now the photon can get through!

Uncertainty

If we are in an eigenstate of the associated operator, we are certain of the outcome of a measurement of a physical quantity Q — it will be the associated eigenvalue.

If not, we can find the average of all the possible outcomes, weighted by their probabilities, and the associated standard deviation. Physicists call these the expectation value $\langle Q\rangle_\psi$ and the uncertainty ΔQ_ψ respectively.

But what if we would like to know about two physical quantities? If we would like to know both with certainty, we would need to be in an eigenstate of both operators at once.

Theorem: We can find a single basis of "simultaneous" eigenstates of $\hat{A}$ and $\hat{B}$ if and only if $\hat{A}\hat{B} = \hat{B}\hat{A}$.

Noncommutativity gives uncertainty!

More on Uncertainty

The inner product obeys the Schwarz inequality

$$|\langle\phi|\psi\rangle|^2 \leq \langle\psi|\psi\rangle\langle\phi|\phi\rangle$$

For appropriate choices of $|\phi\rangle$ and $|\psi\rangle$, a little algebra gives

$$\Delta A_\psi \cdot \Delta B_\psi \geq \frac{1}{2}|\langle\psi|\left(\widehat{A}\widehat{B} - \widehat{B}\widehat{A}\right)|\psi\rangle|$$

When the corresponding operators don't commute, there is a limit on how well we can know two physical quantities at once, because we have a clash of the two different bases of eigenstates we would need to work in to know each one with certainty.

Let's see how this works for the famous example. Now our vectorspace will consist of square-integrable (L^2) functions. . .

An Infinite-Dimensional Vectorspace

A particle moving on a line is specified by a wavefunction $\psi(x)$. Or by its Fourier transform, $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x)\, dx$.

- $\psi(x)$ gives the state's coordinates in the position basis (specifies the function by its graph)
- $\tilde{\psi}(p)$ gives the state's coordinates in the momentum basis (specifies the function by its frequency spectrum)

Define the inner product on this vectorspace:

$$\langle\phi|\psi\rangle = \int_{-\infty}^{\infty} \phi(x)^* \psi(x) dx = \int_{-\infty}^{\infty} \tilde{\phi}(p)^* \tilde{\psi}(p) dp$$

Two linear operators on this vectorspace:

- $\widehat{x}: \ \psi(x) \to x\psi(x)$ or $\tilde{\psi}(p) \to i\hbar \frac{d}{dp} \tilde{\psi}(p)$.
- $\widehat{p}: \ \psi(x) \to -i\hbar \frac{d}{dx} \psi(x)$ or $\tilde{\psi}(p) \to p\tilde{\psi}(p)$.

Infinite-Dimensional Uncertainty

The commutator of our two operators:

$$(\widehat{x}\widehat{p} - \widehat{p}\widehat{x}): \;\; \psi(x) \to x\left(-i\hbar\frac{d}{dx}\psi(x)\right) - \left(-i\hbar\frac{d}{dx}\left(x\psi(x)\right)\right) = i\hbar\psi(x)$$

So $(\widehat{x}\widehat{p} - \widehat{p}\widehat{x}) = i\hbar \cdot \widehat{\mathbb{1}}$ and the uncertainty principle becomes

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

Just the mathematical statement that a sharp spike requires a lot of frequencies, or conversely a wave with a narrow range of frequencies must be spread widely throughout space.

This is why the sharp spikes in a high-speed data connection require a lot of bandwidth (many frequencies).

Theorem: It is only possible to have $(\widehat{x}\widehat{p} - \widehat{p}\widehat{x}) = i\hbar \cdot \widehat{\mathbb{1}}$ in an infinite-dimensional vector space.

Proof (one line) is left to the reader.

Entangling Alliances

If we have two photons, our states are now $|\updownarrow\rangle|\updownarrow\rangle$, $|\updownarrow\rangle|\leftrightarrow\rangle$, etc.

Dimension of the space multiplies (tensor product).

Then we can have entangled states:

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|\updownarrow\rangle|\leftrightarrow\rangle + |\leftrightarrow\rangle|\updownarrow\rangle)$$

Each photon has a 50% probability of being horizontal, a 50% probability of being vertical. But whatever we measure for the first photon, the second is always opposite!

Often applied to quantum spin, which works the same way with $|\updownarrow\rangle \Rightarrow |\uparrow\rangle$ (spin up) and $|\leftrightarrow\rangle \Rightarrow |\downarrow\rangle$ (spin down). Note that orthogonal quantum states are now 180° apart in the real world. (We have replaced $SO(3)$ by $SU(2)$.)

Useful for quantum computation, quantum encryption, and more...

For more information:

`http://community.middlebury.edu/~ngraham/index.html#linearalgebra`