# The Scattering Theory Approach to the Casimir Energy

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# Credits

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# Objective

I will describe an efficient machinery for computing Casimir interaction energies for a wide range of object configurations. Just:

- Pick an appropriate basis for each object
- Describe each object individually through its scattering data in that basis
- Describe the relative positions and orientations of the objects through standard change of basis formulae (e.g. expansion of a plane wave in spherical waves)

The method provides both analytic expansions at large separations and numerical results for arbitrary separation, in both cases without the need for large-scale calculations or computations.

I will focus especially on geometries involving edges and tips.

# Background

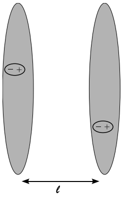
This approach builds on a range of prior work:

- Asymptotic multiple scattering using surface scattering kernel
  Balian, Duplantier
- $\mathbb{T}$-operator methods for proving general theorems
  Kenneth, Klich
- Scattering theory approach for parallel plates, Lifshitz theory
  Kats; Renne; Genet, Jaekel, Lambrecht, Maia Neto, Reynaud
- Many-body $\mathbb{S}$-matrix techniques for disks and spheres
  Bulgac, Henlseler, Magierski, Wirzba
- Path integral Casimir techniques
  Bordag, Robaschik, Scharnhorst, Wieczorek
  Emig, Golestanian, Hanke, Kardar
- Scattering theory Casimir methods for single bodies
  Jaffe, Khemani, Graham, Quandt, Scandurra, Weigel

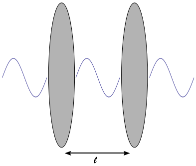
# Quantum Fluctuations

Two complementary pictures of Casimir interactions:

Charge fluctuations: We have induced dipole-dipole interactions. When we sum over all possible fluctuations, energy is lowered by dipole-dipole attraction (plus contributions from higher multipoles).




Field fluctuations: Each mode of the electromagnetic field carries energy $E = \hbar\omega(n + \frac{1}{2})$, where $n$ is the number of photons in that mode. So even if $n = 0$, we have energy $\frac{1}{2}\hbar\omega$. Moving the plates changes the allowed spectrum of modes, thereby altering the sum over all modes of this "zero-point" energy.




The key point: An object is completely represented by its electromagnetic response.

# The Method

We start from the electromagnetic path integral in $A_0 = 0$ gauge.
Decompose as Fourier series, with frequency $\omega$ and time interval $T$:

$$Z = \prod_\omega \int \mathcal{D}\mathbf{A} \exp\left[\frac{iT}{2\hbar}\int d\mathbf{x}\, \mathbf{E}(\omega,\mathbf{x})^\dagger \left(\mathbb{H}_0(\omega) - \frac{\mathbb{V}(\omega,\mathbf{x})}{k^2}\right)\mathbf{E}(\omega,\mathbf{x})\right]$$

with $\mathbb{H}_0(\omega) = \left(\mathbb{I} - \frac{1}{k^2}\nabla\times\nabla\times\right)$, $\omega = ck$. The interaction is

$$\mathbb{V}(\omega,\mathbf{x}) = \mathbb{I}k^2(1-\epsilon(\omega,\mathbf{x})) + \nabla\times\left(\mu(\omega,\mathbf{x})^{-1} - 1\right)\nabla\times$$

Use Hubbard-Stratonovich: multiply and divide by

$$W = \prod_\omega \int \mathcal{D}\mathbf{J} \exp\left[\frac{iT}{2\hbar}\int d\mathbf{x}\, \mathbf{J}(\omega,\mathbf{x})^\dagger \mathbb{V}^{-1}(\omega,\mathbf{x})\mathbf{J}(\omega,\mathbf{x})\right] = \sqrt{\det\mathbb{V}}$$

Then shift variables, using the free Green's function $\mathbb{G}_0(\omega,\mathbf{x},\mathbf{x}')$,

$$\begin{aligned}\mathbf{J}'(\omega,\mathbf{x}) &= \mathbf{J}(\omega,\mathbf{x}) + \frac{1}{k}\mathbb{V}(\omega,\mathbf{x})\mathbf{E}(\omega,\mathbf{x})\\ \mathbf{E}'(\omega,\mathbf{x}) &= \mathbf{E}(\omega,\mathbf{x}) + k\int d\mathbf{x}'\mathbb{G}_0(\omega,\mathbf{x},\mathbf{x}')\mathbf{J}'(\omega,\mathbf{x}')\end{aligned}$$

where $-k^2\mathbb{H}_0(\omega)\mathbb{G}_0(\omega,\mathbf{x},\mathbf{x}') = \mathbb{I}\delta^{(3)}(\mathbf{x}-\mathbf{x}')$.

# Going to the source

After Hubbard-Stratonovich, the path integral in $\mathbf{E}'$ is that of a free field,

$$Z_0 = \prod_{\omega} \int \mathcal{D}\mathbf{A}' \exp\left[\frac{iT}{2\hbar} \int d\mathbf{x}\, \mathbf{E}'(\omega, \mathbf{x})^{\dagger} \mathbb{H}_0(\omega) \mathbf{E}'(\omega, \mathbf{x})\right]$$

We have traded interactions of the fields $\mathbf{E}$ for interactions of the sources $\mathbf{J}'$, which are restricted to the objects,

$$\begin{aligned} Z \;=\; & \frac{Z_0}{W} \prod_{\omega} \int \mathcal{D}\mathbf{J}' \exp\left[\frac{iT}{2\hbar} \int d\mathbf{x}\, d\mathbf{x}' \right. \\ & \left. \mathbf{J}'(\omega, \mathbf{x}')^{\dagger} \left(\mathbb{G}_0(\omega, \mathbf{x}, \mathbf{x}') + \mathbb{V}^{-1}(\omega, \mathbf{x}) \delta^{(3)}(\mathbf{x} - \mathbf{x}')\right) \mathbf{J}'(\omega, \mathbf{x})\right] \end{aligned}$$

Both $W = \sqrt{\det \mathbb{V}}$ and $Z_0 = \sqrt{\det \mathbb{H}_0^{-1}}$ are local and do not depend on the separations of the objects. These terms contain all (renormalized) divergences. As long as we are comparing two configurations, not changing the objects themselves, this contribution cancels and can be ignored.

# Meet Mr. T

From the partition function, we get the Casimir energy

$$\mathcal{E} = \frac{i\hbar}{T} \log Z = \frac{i\hbar}{2T} \log \det(\mathbb{G}_0 + \mathbb{V}^{-1})^{-1} + \mathcal{E}_0$$

where $\mathcal{E}_0$ is a constant, independent of the objects' positions, and $\mathbb{T} = (\mathbb{G}_0 + \mathbb{V}^{-1})^{-1} = \mathbb{V}(\mathbb{I} + \mathbb{G}_0\mathbb{V})^{-1}$ is the $\mathbb{T}$-operator:

- Connects different values of **k** ("off-shell").
- Proportional to $\mathbb{V}$, so $\langle \mathbf{x}|\mathbb{T}|\mathbf{x}'\rangle = 0$ if $\mathbf{x}$ or $\mathbf{x}'$ is not on an object.

The strategy: Decompose $\det \mathbb{T}^{-1} = \det(\mathbb{G}_0 + \mathbb{V}^{-1})$ using a position space basis (which is restricted to points on the objects, since otherwise **J** vanishes) divided into blocks, where each block is labeled by the object on which the corresponding points lie.

- The off-diagonal blocks in this expansion only involve $\mathbb{G}_0$.
- The diagonal blocks in this expansion only involve $\mathbb{F}$, the matrix of on-shell scattering amplitudes ($T$-matrix).
- Let $\mathbb{T}_\infty$ be the $\mathbb{T}$-operator for a reference configuration with the objects at infinity. It is block diagonal in this basis.

# The Plan Comes Together

We obtain the energy

$$\mathcal{E} - \mathcal{E}_\infty = \frac{i\hbar}{2\pi} \int_0^\infty d\omega \log \det \left( \mathbb{M}\mathbb{M}_\infty^{-1} \right) , \quad \text{with}$$

$$\mathbb{M} = \begin{pmatrix} (\mathbb{F}^1)^{-1} & \mathbb{U}^{12} & \cdots \\ \mathbb{U}^{21} & (\mathbb{F}^2)^{-1} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \qquad \mathbb{M}_\infty^{-1} = \begin{pmatrix} \mathbb{F}^1 & 0 & \cdots \\ 0 & \mathbb{F}^2 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

We express each block in a multipole (partial wave) basis.

- $\mathbb{U}^{ij}(\omega)$ is the translation matrix, which gives the change of basis from one object to another (independent of the objects). Obtained from expansion of the free Green's function.
- $\mathbb{F}^i(\omega)$ is the matrix of scattering amplitudes (aka $T$-matrix) for each object individually (independent of separation). Obtained from on-shell scattering.

For two objects: $\det \left( \mathbb{M}\mathbb{M}_\infty^{-1} \right) = \det \left( \mathbb{I} - \mathbb{F}^1 \mathbb{U}^{12} \mathbb{F}^2 \mathbb{U}^{21} \right)$

# Upon Further Reflection

For two objects, we have found

$$\mathcal{E} - \mathcal{E}_\infty = \frac{i\hbar}{2\pi} \int_0^\infty d\omega \log\det\left(\mathbb{I} - \mathbb{N}\right)$$

where $\mathbb{N} = \mathbb{F}^1\mathbb{U}^{12}\mathbb{F}^2\mathbb{U}^{21}$. In some cases, we will use this form directly. It can also be convenient to write

$$\log\det\left(\mathbb{I} - \mathbb{N}\right) = \mathrm{tr}\log\left(\mathbb{I} - \mathbb{N}\right) = -\,\mathrm{tr}\left(\mathbb{N} + \frac{\mathbb{N}^2}{2} + \frac{\mathbb{N}^3}{3} + \dots\right)$$

which puts our expression in the form of a multiple reflection expansion, where $\mathbb{N}$ represents a single reflection (back and forth).

- This expansion is particularly useful for cases where $\mathbb{N}$ is given in a continuous basis.
- For parallel plates, the expansion of $\zeta(4) = 1 + \frac{1}{16} + \frac{1}{81} + \dots$ gives the multiple reflection expansion, and provides an estimate of its convergence (in the worst case).

# The Ingredients: 1. Scattering Bases

For each object individually, we choose a standard basis in which we can write down the eigenfunctions of the free vector Helmholtz equation (e.g. Cartesian, spherical, etc.). These are the regular solutions $|\mathbf{E}^{\text{reg}}_{\alpha}(\omega)\rangle$. We also have the outgoing free solutions $|\mathbf{E}^{\text{out}}_{\alpha}(\omega)\rangle$, which are independent of the regular solutions and typically singular at the origin.

We will need textbook results for the free Green's function and the expansion of a plane wave in terms of spherical Bessel functions and vector spherical harmonics:

$$\mathbb{G}_0(\mathbf{x}_1, \mathbf{x}_2, k) = ik \sum_{\ell jm} j_\ell(kr_<) h^{(1)}_\ell(kr_>) \mathbf{Y}^\ell_{jm}(\theta_1, \phi_1)^* \otimes \mathbf{Y}^\ell_{jm}(\theta_2, \phi_2)$$

$$\boldsymbol{\xi} e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi \sum_{\ell jm} i^\ell \left(\boldsymbol{\xi} \cdot \mathbf{Y}^\ell_{jm}(\theta_k, \phi_k)^*\right) \underbrace{j_\ell(kr) \mathbf{Y}^\ell_{jm}(\theta, \phi)}_{\mathbf{E}^{\text{reg}}_{\alpha}(\omega, \mathbf{x})}$$

# The Ingredients: 2. Scattering Amplitudes

Lippman-Schwinger equation for full scattering solution $\mathbf{E}_\alpha(\omega, \mathbf{x})$:

$$\mathbf{E}_\alpha(\omega, \mathbf{x}) = \mathbf{E}_\alpha^{\text{reg}}(\omega, \mathbf{x}) - \mathbb{G}_0 \mathbb{V} \mathbf{E}_\alpha(\omega, \mathbf{x}) = \mathbf{E}_\alpha^{\text{reg}}(\omega, \mathbf{x}) - \mathbb{G}_0 \mathbb{T} \mathbf{E}_\alpha^{\text{reg}}(\omega, \mathbf{x})$$

Use the free Green's function

$$\mathbb{G}_0(\omega, \mathbf{x}, \mathbf{x}') = \sum_\alpha \mathbf{E}_\alpha^{\text{reg}}(\omega, \mathbf{x}_<) \otimes \mathbf{E}_\alpha^{\text{out}}(\omega, \mathbf{x}_>)$$

to express the full solution far away from the object as a linear combination of regular and outgoing waves:

$$\mathbf{E}_\alpha(\omega, \mathbf{x}) = \mathbf{E}_\alpha^{\text{reg}}(\omega, \mathbf{x}) - \sum_\beta \mathbf{E}_\beta^{\text{out}}(\omega, \mathbf{x}) \underbrace{\int d\mathbf{x}' \, \mathbf{E}_\beta^{\text{reg}}(\omega, \mathbf{x}')^\dagger \mathbb{T} \mathbf{E}_\alpha^{\text{reg}}(\omega, \mathbf{x}')}_{\mathbb{F}_{\beta\alpha}(\omega)}$$

For the diagonal blocks, we will need this matrix of scattering amplitudes in our chosen basis (from a calculation or measurement).

# The Ingredients: 3. Translation Matrices

For the off-diagonal blocks, we need the translation matrices, which decompose the free Green's function in that basis and the corresponding expansion of a plane wave. For for $|\mathbf{x}| < |\mathbf{x}'|$ we have

$$\mathbb{G}_0(\omega, \mathbf{x}, \mathbf{x}') = \sum_{\alpha} \mathbf{E}^{\text{reg}}_{\alpha}(\omega, \mathbf{x}) \otimes \mathbf{E}^{\text{out}}_{\alpha}(\omega, \mathbf{x}')$$

The translation matrix $\mathbb{U}^{ji}$ gives the expansion of an outgoing wave from object $i$ in terms of regular waves for object $j$,

$$\mathbf{E}^{\text{out}}_{\alpha}(\omega, \mathbf{x}_i) = \sum_{\beta} \mathbb{U}^{ji}_{\beta\alpha}(\omega) \mathbf{E}^{\text{reg}}_{\beta}(\omega, \mathbf{x}_j)$$

The free Green's function becomes

$$\mathbb{G}_0(\omega, \mathbf{x}, \mathbf{x}') = \sum_{\alpha,\beta} \mathbf{E}^{\text{reg}}_{\alpha}(\omega, \mathbf{x}_i) \otimes \mathbb{U}^{ji}_{\alpha\beta}(\omega) \mathbf{E}^{\text{reg}}_{\beta}(\omega, \mathbf{x}'_j)$$

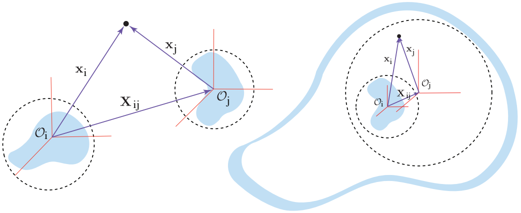
where $\mathbf{x}_i$ is in the coordinates for object $i$ and $\mathbf{x}'_j$ is in the coordinates for object $j$. The bases for different objects can be chosen differently (e.g. spherical, cylindrical, Cartesian).

# Limitations

The scattering expansion identified a “radial” coordinate for each object, in order to define regular and outgoing waves. This identification must be the same across all points on each object.

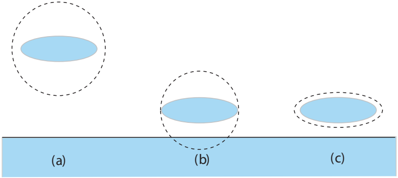
- All points on the object must have a smaller value of this radial coordinate than any point on another object. Or, the object can always have a larger value of the radial coordinate (if one object is inside the other).

ex. Spherical basis for both objects:



# Avoiding Limitations

ex. Elliptic cylinder and plane:




(a) Objects separated by ordinary cylinder:
ordinary cylindrical coordinates OK

(b) Enclosing ordinary cylinder intersects plane:
ordinary cylindrical coordinates FAIL

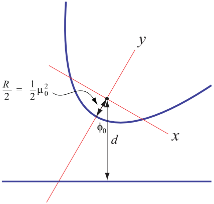
(c) Objects separated by elliptic cylinder:
elliptic cylindrical coordinates OK

In some geometries (particularly for close separations), another basis may be more convenient, such as a spatially localized position basis.

# Applications: Parabolic Cylinder

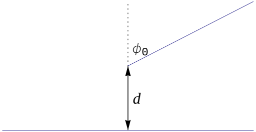
For smooth surfaces at small separations, the PFA + derivative expansion provides a valuable tool. Fosco, Lombardo, Mazzitelli

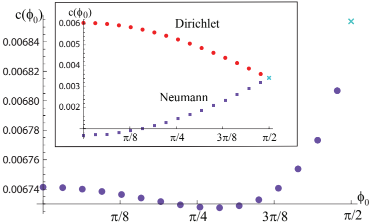
We will focus on geometries with tips and edges, where this expansion is often invalid.
Parabolic cylinder geometry is separable and gives a half-plane as a limiting case.
Electromagnetic result is the sum of Dirichlet and Neumann contributions.




Half-plane opposite a plane, as a function of angle:

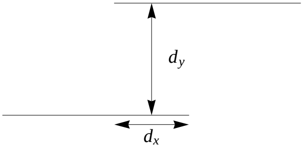
$$\frac{\mathcal{E}}{L} = -\frac{\hbar c}{d^2 \cos\phi_0} \cdot c(\phi_0)$$

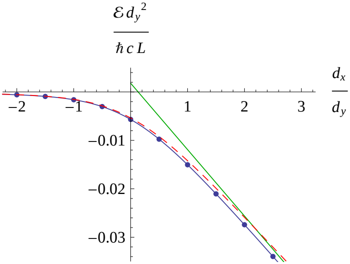


# Results: Parabolic Cylinder

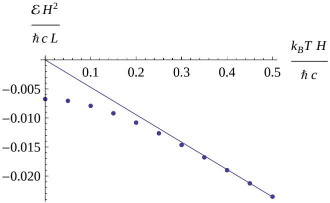
Overlapping planes, as a function of overlap:




Straight line represents PFA + edge correction, dashed line is analytic approximation based on the first reflection.

Half-plane perpendicular to a plane, as a function of temperature (solid line is $T \to \infty$ result):




Can compare to other edge geometry methods. Kabat, Karabali, Nair

Gies, Klingmuller, Weber

# Results: Sample Analytic Expressions

The exact Casimir interaction energy for a half-plane perpendicular to a plane:

$$-\frac{\mathcal{E} d^2}{\hbar c L} = -\int_0^\infty \frac{q\,dq}{4\pi} \log\det\left(\delta_{\nu\nu'} - (-1)^\nu \mathrm{k}_{-\nu-\nu'-1}(2q)\right)$$

where $\mathrm{k}_\ell(u)$ is the Bateman k-function.

The Casimir interaction energy for parallel planes overlapping by $d_x$, at first order in the reflection expansion:

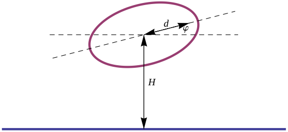
$$-\frac{\mathcal{E} d^2}{\hbar c L} = \frac{1}{24\pi^3}\left[\frac{d^2}{d^2+d_x^2} + 3\left(1 - i\frac{d_x}{d}\log\frac{id-d_x}{\sqrt{d^2+d_x^2}}\right)\right] + \cdots$$

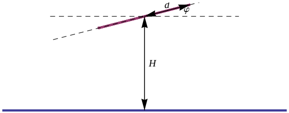
The Casimir interaction energy for a half-plane tilted by angle $\phi_0$ opposite a plane, at second order in the reflection expansion:

$$-\frac{\mathcal{E} d^2}{\hbar c L} = \frac{\sec\phi_0}{16\pi^2} + \frac{1}{256\pi^3}\left(\frac{4}{3} + \csc^3\phi_0 \sec\phi_0 (2\phi_0 - \sin 2\phi_0)\right) + \cdots$$

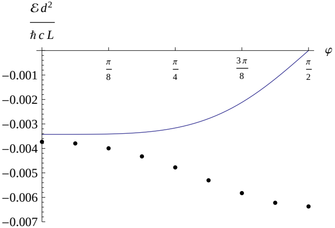
# Applications: Elliptic Cylinder

Elliptic cylinder geometry allows us to study a strip as its zero radius limit:




The strip prefers a perpendicular orientation:




Graph shows the case $H = 2d$. PFA (solid line) goes to zero as $\varphi \to \frac{\pi}{2}$. Derivative expansion correction makes this estimate worse!

Compare $\varphi = \frac{\pi}{2}$ to a half-plane: Magnitude of the energy is slightly smaller, as expected, but larger than would be obtained by writing the energy of the strip as the difference of two half-planes at different distances — shows non-superposition effects.

# Applications: Wedge and Cone

We can also handle a wedge and cone by treating $\theta$ as the "radial" variable. Requires analytic continuation in the angular momentum.

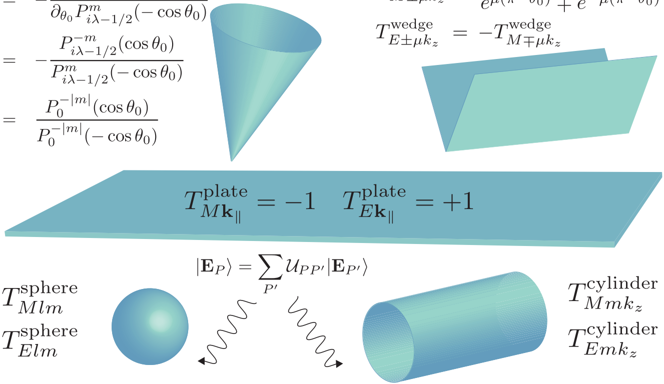
$$T^{\text{cone}}_{M\,\lambda m} = -\frac{\partial_{\theta_0} P^{-m}_{i\lambda-1/2}(\cos\theta_0)}{\partial_{\theta_0} P^{m}_{i\lambda-1/2}(-\cos\theta_0)}$$

$$T^{\text{cone}}_{E\,\lambda m} = -\frac{P^{-m}_{i\lambda-1/2}(\cos\theta_0)}{P^{m}_{i\lambda-1/2}(-\cos\theta_0)}$$

$$T^{\text{cone}}_{Gh\,m} = \frac{P^{-|m|}_{0}(\cos\theta_0)}{P^{-|m|}_{0}(-\cos\theta_0)}$$

$$T^{\text{wedge}}_{M\pm\mu k_z} = \frac{e^{\mu\theta_0} \mp e^{-\mu\theta_0}}{e^{\mu(\pi-\theta_0)} \mp e^{-\mu(\pi-\theta_0)}}$$

$$T^{\text{wedge}}_{E\pm\mu k_z} = -T^{\text{wedge}}_{M\mp\mu k_z}$$



$$T^{\text{plate}}_{M\mathbf{k}_\parallel} = -1 \qquad T^{\text{plate}}_{E\mathbf{k}_\parallel} = +1$$

$$|\mathbf{E}_P\rangle = \sum_{P'} \mathcal{U}_{PP'} |\mathbf{E}_{P'}\rangle$$

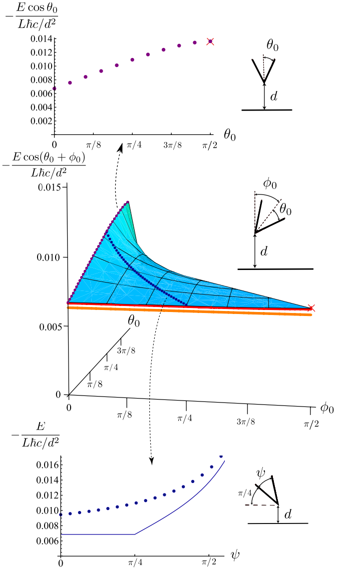
$$T^{\text{sphere}}_{Mlm} \qquad T^{\text{sphere}}_{Elm} \qquad T^{\text{cylinder}}_{Mmk_z} \qquad T^{\text{cylinder}}_{Emk_z}$$

For the cone, introduce a ghost polarization to cancel $\ell = 0$ mode.

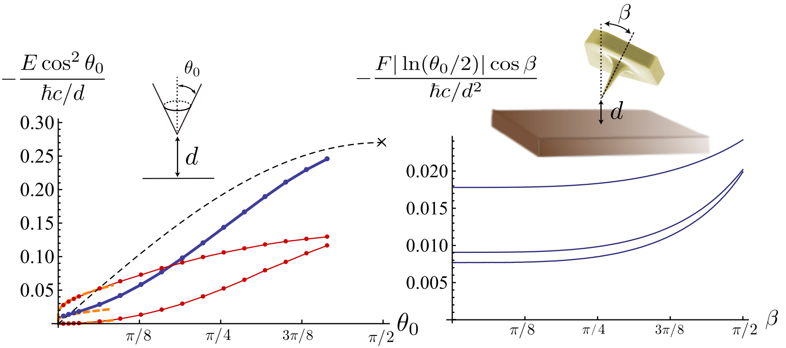
# Results: Wedge

Casimir interaction energy of a wedge at distance $d$ above a plane, as a function of its semi-opening angle $\theta_0$ and tilt $\phi_0$, using a multiple reflection expansion.

The energy as a function of $\theta_0$ and $\phi_0$ is shown in the middle figure. The symmetric case, $\phi_0 = 0$, is displayed at the top. The case where the back side of the wedge is "hidden" from the plane is shown at the bottom, with a comparison to the PFA.



# Results: Cone




Casimir interaction energy of a cone of semi-opening angle $\theta_0$ a distance $d$ above a plane. In the left figure, the cone is oriented vertically, with the energy multiplied by $\cos^2\theta_0$ to remove the divergence as $\theta_0 \to \pi/2$. The right figure shows the force, suitably scaled, for a tilted, sharp cone ($\theta_0 \to 0$, evocative of an AFM tip) as a function of tilt angle $\beta$ for temperatures $\tau$=300 K, 80 K, and 0 K (top to bottom), at a separation of 1 $\mu$m.

# Further Extensions

We and other groups have applied and extended these methods:

- Related $\mathbb{T}$-operator techniques — Kenneth, Klich
- Exact results for dilute systems — Milton, Parashar, Wagner
- Position basis techniques — Johnson, Reid, Rodriguez, White
- Non-superposition effects — Fosco, Losada, Ttira; Emig, Rahi, Rodriguez-Lopez
- Lifshitz theory perturbation expansion — Golestanian
- Corrugated surfaces — Cavero-Pelaez, Milton, Parashar, Shajesh
- Objects inside one another — Emig, Jaffe, Kardar, Rahi, Zaheer
- Casimir Earnshaw's Theorem — Emig, Kardar, Rahi
- Casimir Babinet Principle — Abravanel, Jaffe, Maghrebi
- Dynamical Casimir Effects — Golestanian, Kardar, Maghrebi
- Intersecting objects — Schaden
- Non-equilibrium Casimir forces — Bimonte, Emig, Kardar, Krüger
- Techniques for computing general $T$-matrices — Forrow, Graham