A rigorous Theory of the Diffraction of electromagnetic Waves on a perfectly conducting Disk

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The problem of the diffraction of electromagnetic waves on the perfectly conducting disk is severely under-researched. The diffraction through the circular opening in a perfectly conducting, infinite plane lets us reduce the problem by means of a given generalization in Babinet's Principle

1 Formulating the Diffraction Problem

In the theory of electromagnetic wave diffraction ultimately lies the following problem. A given incident wave comes upon a barrier, a diffracting object, and it is to seek an outgoing wave, which at great distance acts as a spherical wave with distance-dependent amplitude, and which adds to the incident wave, which satisfies the boundary conditions. The nature of these boundary conditions depends upon which features the diffracting object has, whether it is black and absorbs all radiation that lands on it, or whether, for example, it is perfectly conducting, that is to say, the radiation interacting with the object is fully emitted and so forth.

We concern ourselves in the following with a planar screen of finite dilation, particularly with disks as diffraction objects, by which we assume, that it is infinitely thin and perfectly conducting. Physically speaking, that its thickness is small compared to the wavelength, but still large enough that the screen is opaque. The requirement of perfect conductivity is adequately met by the usual metals in the regions of the actual electromagnetic waves (wavelength > 1mm), is as demonstrated by their reflection coefficient.

With this diffraction problem there is always the complementary problem, namely the diffraction and reflection of an electromagnetic wave through the opening of a finite dilation in an infinite, perfectly conducting plane. This follows from what has been proved in the next section, Babinet's general Principle.

In the case of planar screens, regarding the particular conditions of the electromagnetic waves, lies something obverse to diffraction in a sphere, because the edge of the screen causes a singularity of the electromagnetic fields. This occurs already in the static limit of a perfectly conducting, planar screen in the homogenous electric field phenomenon, if the screen is not vertical to the force lines; there is in fact a finite electric dipole moment induced in the screen, but the influential charge density would be infinitely high at the edges of the screen and as such also the strength of the electric field.

With the scalar diffraction problem for planar screens, as is in acoustics, the diffraction wave remains finite on the edges. This fact is incurred in the requirements and shows that the scalar diffraction problem is distinctly solvable. The electromagnetic diffraction problem, in contrast, substitutes and circumvents the claim of the finiteness of the diffraction wave with another claim. It is actually required for the sufficient enforcement of the unique resolvability of the problem that its energy in each finite area is finite. The electric and magnetic field strengths around the borders of the screen may be infinitely large, but only so that they are squarely integrable. As this requirement is fulfilled, only the incident to the screen energy of the emitted wave, and no additional energy, is radiated; if it are not fulfilled, then we do not have an absolute diffraction problem. Rather the electromagnetic field of the clean diffraction is overlaid by additional radiation, which takes place at the screen edges and can be viewed as a kind of driven radiation.

The problem of the diffraction of an incident electromagnetic wave $\mathfrak{E}^i$, $\mathfrak{B}^i$ with the time dependence $e^{i\,\omega\,t}$ on the perfectly conducting, planar screen of a finite dilation is thus formulated. There is a diffracted wave $\mathfrak{E}^r$, $\mathfrak{B}^r$ with a matching time component and with the following conditions:

1. The field $\mathfrak{E}^\mathfrak{r}, \mathfrak{B}^\mathfrak{r}$, like $\mathfrak{E}^\mathfrak{i}, \mathfrak{B}^\mathfrak{i}$ adhere to Maxwell's equations:

$$\text{curl } \mathfrak{E} = -i\omega\mu\mathfrak{B}; \qquad \text{curl } \mathfrak{B} = +i\omega\varepsilon\mathfrak{E} \tag{1}$$

At all space with the dielectric constant ε and permeability constant μ, which are both presumed to be location-independent.

2. At large distance from the screen, the the diffracted wave relates to an outgoing spherical wave with directional amplitude; that is to say, it satisfies the Sommerfeld radiation condition[1].

3. On the diffracting screen (considering it is perfectly conducting):

$$(\mathfrak{E}^i + \mathfrak{E}^r)_{tang} = 0 \tag{2}$$

4. On the edge of the diffracting screen, $\mathfrak{B}^r$ and $\mathfrak{B}^i$ would be infinite in such a way that their electromagnetic energy density is integrable.

Attempts to treat the diffracting electromagnetic waves on a circular disk that have previously been made yield a solution that fits conditions 1-3, but not 4[2]. In the following, a rigorous solution to this problem will be developed.

[1] A. Sommerfeld, Jber. dtsch. Math. Ver. 21, 309 (1913)
[2] S. Anm. 5.

2 Babinet's General Principle

We compare the following:

1. The diffraction of an incident wave, not necessarily planar, on a planar, perfectly conducting screen F, which is everywhere finite;

2. The diffraction of an incident wave on a planar, perfectly conducting screen F', which complements the screen F.

The screens F, F' are complementary. The plane in which they lie shall be the plane z = 0.

The initial conditions of perfect conductivity claim, for the global field strengths $\mathfrak{E}, \mathfrak{B}$

$$\mathfrak{E}_x = \mathfrak{E}_y = 0, \quad \frac{\partial \mathfrak{E}_z}{\partial z} = 0, \quad \frac{\partial \mathfrak{B}_x}{\partial z} = \frac{\partial \mathfrak{B}_y}{\partial z} = 0, \quad \mathfrak{B}_z = 0 \tag{3}$$

They apply identically to all x, y of the perfectly conductive area where $z = 0$. The first two are equivalent to (2), the last four follow from Maxwell's equations (1) from the first two. We now require the three axioms:

1. If $\mathfrak{E}$ and $\mathfrak{B}$ with the arguments x, y, z a solution of the Maxwell's equations (1), then through the components $\mathfrak{E}_x, \mathfrak{E}_y, -\mathfrak{E}_z, -\mathfrak{B}_x, -\mathfrak{B}_y, \mathfrak{B}_z$, with the arguments x, y, z an electromagnetic field that is also a solution to Maxwell's equations is given.

2. If $\mathfrak{E}, \mathfrak{B}$ is the electromagnetic field of a planar electromagnetic wave radiation, then $\mathfrak{E}_x, \mathfrak{E}_y, \mathfrak{B}_z$ are symmetrical to the plane of the figure with $\mathfrak{E}_z, \mathfrak{B}_x, \mathfrak{B}_y$, with opposite values.

3. If $\mathfrak{E}, \mathfrak{B}$ a solution to Maxwell's equations (1), the following field is as well:

$$\mathfrak{E}' = \pm\sqrt{\frac{\mu}{\varepsilon}}\mathfrak{B}, \qquad \mathfrak{B}' = \mp\sqrt{\frac{\varepsilon}{\mu}}\mathfrak{E}$$

The first is easy to demonstrate. The second axiom is yielded when we calculate an arbitrary distribution of electric charge density and current (from which we deduce the electromagnetic field) and the radiated field with the help of the electrodynamic potential. The third is self-evident.

At the diffraction at the finite planar F, the incident, for all z defined waves (Index e) a diffracted wave (index b) is superposed, whose behavior for $z \geq 0$ follows from the second axiom, and which fulfills the radiation conditions. We write the elimination of the arguments x, y:

$$\begin{aligned}
\mathfrak{E}_x &= \mathfrak{E}^i_x(z) + \mathfrak{E}^r_x(z) \quad \text{or} \quad \mathfrak{E}^i_x(z) + \mathfrak{E}^r_x(-z) \\
\mathfrak{E}_y &= \mathfrak{E}^i_y(z) + \mathfrak{E}^r_y(z) \quad \text{or} \quad \mathfrak{E}^i_y(z) + \mathfrak{E}^r_y(-z) \\
\mathfrak{E}_z &= \mathfrak{E}^i_z(z) + \mathfrak{E}^r_z(z) \quad \text{or} \quad \mathfrak{E}^i_z(z) - \mathfrak{E}^r_z(-z) \\
\mathfrak{B}_x &= \mathfrak{B}^i_x(z) + \mathfrak{B}^r_x(z) \quad \text{or} \quad \mathfrak{B}^i_x(z) - \mathfrak{B}^r_x(-z) \\
\mathfrak{B}_y &= \mathfrak{B}^i_y(z) + \mathfrak{B}^r_y(z) \quad \text{or} \quad \mathfrak{B}^i_y(z) - \mathfrak{B}^r_y(-z) \\
\mathfrak{B}_z &= \mathfrak{B}^i_z(z) + \mathfrak{B}^r_z(z) \quad \text{or} \quad \mathfrak{B}^i_z(z) + \mathfrak{B}^r_z(-z)
\end{aligned} \tag{4}$$

The boundary conditions applied to F, at length

$$\text{on F,} \quad \begin{aligned} &\mathfrak{E}^i_x + \mathfrak{E}^r_x = 0, \quad \mathfrak{E}^i_y + \mathfrak{E}^r_y = 0, \quad \mathfrak{B}^i_z + \mathfrak{B}^r_z = 0 \\ &\frac{\partial}{\partial z}(\mathfrak{E}^i_z + \mathfrak{E}^r_z) = 0, \quad \frac{\partial}{\partial z}(\mathfrak{B}^i_x + \mathfrak{B}^r_x) = 0, \quad \frac{\partial}{\partial z}(\mathfrak{E}^i_z + \mathfrak{E}^r_z) = 0 \end{aligned} \tag{5}$$

While for F', the field representations for $z \neq 0$ and its derivatives must continuously cross over into each other; that means:

$$\mathfrak{E}^r_z = 0, \quad \mathfrak{B}^r_x = 0, \quad \mathfrak{B}^r_y = 0, \quad \frac{\partial \mathfrak{E}^r_y}{\partial z} = 0, \quad \frac{\partial \mathfrak{B}^r_z}{\partial z} = 0 \qquad \text{on F'} \tag{6}$$

Moreover, the diffracted wave must fulfill the requirement of integrable electromagnetic energy density.

We claim that the electromagnetic field given by

$$\begin{aligned}
\sqrt{\frac{\varepsilon}{\mu}}\mathfrak{E}'_x &= \mathfrak{B}^i_x(z) - \mathfrak{B}^i_x(-z) + \mathfrak{B}^r_x(z) \quad \text{or} \quad + \mathfrak{B}^r_x(-z) \\
\sqrt{\frac{\varepsilon}{\mu}}\mathfrak{E}'_y &= \mathfrak{B}^i_y(z) - \mathfrak{B}^i_y(-z) + \mathfrak{B}^r_y(z) \quad \text{or} \quad + \mathfrak{B}^r_y(-z) \\
\sqrt{\frac{\varepsilon}{\mu}}\mathfrak{E}'_z &= \mathfrak{B}^i_z(z) + \mathfrak{B}^i_z(-z) + \mathfrak{B}^r_z(z) \quad \text{or} \quad - \mathfrak{B}^r_z(-z) \\
\sqrt{\frac{\mu}{\varepsilon}}\mathfrak{B}'_x &= -\mathfrak{E}^i_x(z) - \mathfrak{E}^i_x(-z) - \mathfrak{E}^r_x(z) \quad \text{or} \quad + \mathfrak{E}^r_x(-z) \\
\sqrt{\frac{\mu}{\varepsilon}}\mathfrak{B}'_y &= -\mathfrak{E}^i_y(z) - \mathfrak{E}^i_y(-z) - \mathfrak{E}^r_y(z) \quad \text{or} \quad + \mathfrak{E}^r_y(-z) \\
\sqrt{\frac{\mu}{\varepsilon}}\mathfrak{B}'_z &= -\mathfrak{E}^i_z(z) + \mathfrak{E}^i_z(-z) - \mathfrak{E}^r_z(z) \quad \text{or} \quad - \mathfrak{E}^r_z(-z)
\end{aligned} \tag{7}$$

for $z \geq 0$ or $z \leq 0$ solves the given complementary electromagnetic diffraction problem for an incident wave with the electric field strength $\sqrt{\frac{\mu}{\varepsilon}}\,\mathfrak{B}^i$ and the magnetic field strength $-\sqrt{\frac{\varepsilon}{\mu}}\,\mathfrak{E}^i$. The proof can be solved in a few lines.

1. For $z \geq 0$, the electromagnetic field is a superposition of the incident wave, which reflects according to the laws of wave reflection, and a diffracted wave, whereas for $z \leq 0$ there only exists a diffracted wave. The diffracted wave meets the radiation conditions for $z \neq 0$, as these do not alter the substitution of the third axiom.

2. All specified waves meet the first and third axioms by way of Maxwell's equations.

3. On the diffracting screen, the boundary conditions (3) apply due to (6).

4. In the opening F, the continuity conditions apply for the electromagnetic field strength and its derivative due to (5).

5. If the electromagnetic field strengths in (4) on the screen's edge is quadratically integrable, we can draw the same conclusion for (7).

Put together, we can say: if the diffracted wave $\mathfrak{E}^r, \mathfrak{B}^r$ is a solution for the diffraction problem for waves on the screen F with the incident wave $\mathfrak{E}^i, \mathfrak{B}^i$, then the waves $+\sqrt{\frac{\mu}{\varepsilon}}\mathfrak{B}^r, -\sqrt{\frac{\varepsilon}{\mu}}\mathfrak{E}^r$ and $-\sqrt{\frac{\mu}{\varepsilon}}\mathfrak{B}^r, +\sqrt{\frac{\varepsilon}{\mu}}\mathfrak{E}^r$ with the incident wave $+\sqrt{\frac{\mu}{\varepsilon}}\mathfrak{B}^i, -\sqrt{\frac{\varepsilon}{\mu}}\mathfrak{E}^i$ are the solution for F' for $z \geq 0$ and $z \leq 0$ respectively. The reversability of this statement is easy to see.

[3]P. Debye, Ann. Phys. (4) 30, 57 [1909].
[4]G. Mie, Ann. Phys (4) 25, 377 [1908].

3 The Debye potentials and their Boundary Constraints

Debye[3] showed, following on Mie's[4] research about the diffraction of electromagnetic waves on the sphere, that the electromagnetic field can be shown to be traversing two scalar potentials Π_1 and Π_2, from which the first is distinguish such that it's radial component of the magnetic vectors disappears, whereas for the second radial component, the radial component of the electric vectors disappears. We mentioned the Debye potentials. They are defined:

$$\mathfrak{E} = \frac{1}{\varepsilon}\text{ curl curl }(\mathfrak{r}\Pi_1) - \frac{i\ k}{\sqrt{\varepsilon\mu}}\text{ curl }(\mathfrak{r}\Pi_2) \tag{8}$$

$$\mathfrak{B} = +\frac{i\ k}{\sqrt{\varepsilon\mu}}\text{ curl }(\mathfrak{r}\Pi_1) + \frac{1}{\mu}\text{ curl curl }(\mathfrak{r}\Pi_2) \tag{9}$$

and satisfy both of the wave equations

$$\nabla^2\Pi_i + k^2\Pi_i = 0 \qquad (i = 1, 2), \tag{10}$$

whereby the wavenumber k

$$k^2 = \omega^2\ \varepsilon\ \mu \tag{11}$$

is given and $\mathfrak{r}$ stands for the radial vector.

The boundary constraints for $\mathfrak{E}$ and $\mathfrak{B}$ can be conveniently converted, in the case of the diffracting sphere at simple boundary constraints for the Debye potentials, and the vectorial electromagnetic diffraction problem can be plainly separated into two independent diffraction problems for Π_1 and Π_2.

The two potentials can be successfuly described when treating the electromagnetic wave diffraction on a perfectly conductive planar screen, particularly when applied to the circular disk and the circular dilation in the infinite plane. Indeed, the question of the boundary constraints brings certian difficulties, but let us resolve them.

The arbitrarily configured screen lies in the $x-y$ plane. Thus, $\mathfrak{E}_x$ and $\mathfrak{E}_y$ must disappear for the entire region of the screen, that is to say by (8) it must hold, as $z=0$

$$\frac{1}{\varepsilon}\left\{\frac{\partial}{\partial x}\left[\Pi_1 + x\frac{\partial \Pi_1}{\partial x} + y\frac{\partial \Pi_1}{\partial y}\right] + k^2\; x\; \Pi_1\right\} + \frac{i\;k}{\sqrt{\varepsilon\;\mu}}\; y\; \frac{\partial \Pi_2}{\partial z} = 0, \tag{12}$$

$$\frac{1}{\varepsilon}\left\{\frac{\partial}{\partial y}\left[\Pi_1 + x\frac{\partial \Pi_1}{\partial x} + y\frac{\partial \Pi_1}{\partial y}\right] + k^2\; y\; \Pi_1\right\} - \frac{i\;k}{\sqrt{\varepsilon\;\mu}}\; x\; \frac{\partial \Pi_2}{\partial z} = 0 \tag{13}$$

These are two partial differential equations in x and y for the functions $\Pi_1$and$\frac{\partial \Pi_2}{\partial z}$. They can be easily integrated through cylindrical coordinates:

$$x = \varrho\,\cos\varphi, \qquad y = \varrho\,\sin\varphi \tag{14}$$

This yields

$$\varrho\,\Pi_1 = \alpha\,(\varphi)\,e^{i\,k\,\varrho} + \beta\,(\varphi)\,e^{-i\,k\,\varrho} \tag{15}$$

$$\sqrt{\frac{\varepsilon}{\mu}}\;\varrho^2\;\frac{\partial \Pi_2}{\partial z} = \frac{d\alpha}{d\varphi}\,e^{i\,k\,\varrho} - \frac{d\beta}{d\varphi}\,e^{-i\,k\,\varrho} \tag{16}$$

for $z=0$ and for all points ϱ,φ on the screen. Here, $\alpha\,(\varphi)$ and $\beta\,(\varphi)$ are both functions which could not determined by the boundary constraints (2); rather, they are derived from the requirement that the electromagnetic energy density be integrable in the vicinity of the screen's edge[5].

4 Formulating the Diffraction Problem with the Debye potentials

Let Π_1^{e} and Π_2^{e} be the Debye potentials of the incident planar wave. They can be represented by closed expressions. The Debye potentials of the diffracting wave shall be Π_1^{b} and Π_2^{b}. For these potentials we propose the following conditions:

1. They obey all wave equations.
2. Π_1^{b}, Π_2^{b} can be described at large distance as outgoing spherical waves.
3. They obey, on the screen, the boundary constraints

$$\begin{aligned}\Pi_1^{\mathrm{e}} + \Pi_1^{\mathrm{b}} &= \alpha\,(\varphi)\,\frac{e^{i\,k\,\varphi}}{\varrho} + \beta\,(\varphi)\frac{e^{-i\,k\,\varrho}}{\varrho},\\ \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{\partial \Pi_2^{\mathrm{e}}}{\partial z} + \frac{\partial \Pi_2^{\mathrm{b}}}{\partial z}\right) &= \frac{d\alpha}{d\varphi}\,\frac{e^{i\,k\,\varrho}}{\varrho^2} - \frac{d\beta}{d\varphi}\,\frac{e^{-i\,k\,\varrho}}{\varrho^2}\end{aligned} \tag{17}$$

4. $\alpha\,(\varphi)$ and $\beta\,(\varphi)$ are to be chosen such that the electromagnetic energy density on the screen edges is integrable.
5. Π_1^{b}, and Π_2^{b} create singularities at the origins of the type described in greater detail in (15) and (16).

It is easy to show, as calculated from the Debye potentials electric field, that this meets our requirements as described in section I. Conversely, it is not difficult to see, that the above requirements for the Debye potentials are essential. Particularly, the finite nature of these potentials follows on the screen edge, the one place where the singularity would be expected, from (15) and (16), provided that the screen edge does not go through the origin. But this can be avoided through suitable choice of constants.

In the following calculation, we shall show that the potentials Π_1^{b}, and Π_2^{b} are true by the aforementioned requirements.

5 Requirements on the Edge in spheroidal Coordinates

To handle diffraction on the disk we use spheroidal coordinates. They are defined by

[5]Solutions of the wave equations which fulfill all requirements except for the one involving integrability can be more readily found. One such solution was found by F. Mglich (Ann. Phys. [4] 83, 609, [1927]). One other, which fulfills the conditions $\Pi_1 = 0$ and $\frac{\partial \Pi_2}{\partial z} = 0$ instead pf (15) and (16), was outlined by the author (Nachr. Akad. Wiss. Gttingen, math. -physik. Kl. 1946, 74). For the last one, Babinet's principle also applies, as was shown also by the author (Z. Naturforschg. 1, 496 [1946]). It is doubtful whether such solutions can be considered a physical reality without integrable energy density.

$$x = c\sqrt{\left(1+\xi^2\right)\left(1-\eta^2\right)}\cos\varphi,$$
$$y = c\sqrt{\left(1+\xi^2\right)\left(1-\eta^2\right)}\sin\varphi, \qquad z = c\,\xi\,\eta \tag{18}$$

and have the variable range

$$0 \le \xi < \infty, \qquad -1 \le \eta \le +1, \qquad 0 \le \varphi \le 2\pi \tag{19}$$

The coordinate surface $\xi = 0$ is a disk on the plane $z = 0$, with a center at the origin and with radius c. It may coincide with the diffracting object. The coordinate surface $\eta = 0$ is the remaining part of the plane $z = 0$.

The volume element in the coordinates ξ, η, φ is given by

$$c^3\left(\xi^2+\eta^2\right)d\xi\,d\eta\,d\varphi \tag{20}$$

From Π_1, Π_2 we may cautiously assume, that $\xi = \eta = 0$ on the edge, so something like a power series in ξ, η can be developed. However, the calculation of the field strength components provides powers of $\left(\xi^2+\eta^2\right)^{-1}$ as factors. With that, the field strength components on the edge become infinitely large.

This circumstance, however, cannot be avoided and lies in the very nature of the subject. Yet the requirement that the electromagnetic field must have an integrable energy density in the area surrounding the edge confines the the order of infinity of the field strength components. We calculate these limitations, while we impose the requirements with the Debye potentials

$$\frac{\partial\Pi_1}{\partial\xi} = 0, \quad \frac{\partial\Pi_1}{\partial\eta} = 0, \quad \frac{\partial\Pi_2}{\partial\xi} = 0, \quad \frac{\partial\Pi_2}{\partial\eta} = 0 \quad \text{for } \xi = \eta = 0 \tag{21}$$

Whereby in a development of Π_1 and Π_2 in powers of ξ and η the linear elements in ξ and η cancel. The proof of (21) can be conducted such that a general power series can be calculated for Π_1 and Π_2, and from it the electromagnetic field strength and where all components that give rise to an infinite field energy go to zero.

6 The Spheroid Function

The wave equations can be separated in the coordinates ξ, η, φ. We have to differentiate between such separated wave functions, which are confined throughout space and clearly behave as standing waves, and such waves which behave as outgoing spherical waves at infinity. We mark the first with:

$$L_n^{m\,(1)}(\xi,\,\eta,\,\varphi;\,i\,\gamma) = S_n^{m\,(1)}(-i\,\xi;\,\gamma)\,Sp_n^m(\eta;\,i\,\gamma)\,e^{i\,m\,\varphi} \tag{22}$$

and the last with

$$L_n^{m\,(4)}(\xi,\,\eta,\,\varphi;\,i\,\gamma) = S_n^{m\,(4)}(-i\,\xi;\,\gamma)\,Sp_n^m(\eta;\,i\,\gamma)\,e^{i\,m\,\varphi} \tag{23a}$$

or

$$L_n^{m\,(3)}(\xi,\,\eta,\,\varphi;\,i\,\gamma) = S_n^{m\,(3)}(-i\,\xi;\,\gamma)\,Sp_n^m(\eta;\,i\,\gamma)\,e^{i\,m\,\varphi} \tag{23b}$$

where

$$\gamma = k\,c \tag{24}$$

The functions L_n^m are the Lamé wave functions. The functions $S_n^{m\,(i)}$ and Sp_n^m we present as spheroidal functions. The functions $Sp_n^m(\eta;\,i\,\gamma)$ are generalizations of the spherical functions $P_n^m(\eta)$, in which $\gamma = 0$ is omitted; they obey the differential equation

$$\frac{d}{d\eta}\left[\left(1-\eta^2\right)\frac{dSp}{d\eta}\right] + \left[\lambda + \gamma^2\,\eta^2 - \frac{m^2}{1-\eta^2}\right] Sp(\eta;\,i\,\gamma) = 0. \tag{25}$$

The functions $S_n^{m\,(i)}$ $(i = 1,\,3,\,4)$ solve, when considered as functions of $-i\,\xi$, the above differential equation. For the separation of the wave equation in the coordinates ξ, η, φ we compare with Magnus and Oberhettinger[6]. The theory and designation of the spheroid functions is drawn from the works of the author, which may appear elsewhere.

[6]W. Magnus a. F. Oberhettinger, Form. and Theo. for spec. func. in Math. Phys. Berlin 1943

In order that the wave functions (22) and (23) be unambiguous, m must be a whole number. The separation parameter determines that $Sp_n^m\,(\eta;\,i\,\gamma)$ in $\eta = \pm 1$ remain finite. There is a sequence of such values, which we mark with $\lambda_{|m|}^m\,(i\,\gamma)$, $\lambda_{|m|+1}^m\,(i\,\gamma)$, $\lambda_{|m|+2}^m\,(i\,\gamma)$... or generally $\lambda_n^m\,(i\,\gamma)$. n is also a whole, nonnegative number. For small γ

$$\lambda_{|m|}^m\,(i\,\gamma) = n(n+1) - \gamma^2\,\frac{2\,n^2 + 2\,n - 2\,m^2 - 1}{(2\,n+3)\,(2\,n-1)} + \mathcal{O}\,(\gamma^4) \tag{26}$$

For the spheroid functions, the following is established by Niven[7]:

$$Sp_n^m\,(\eta;\,i\,\gamma) = \sum_{r \geq m-n} i^r\,a_{n,\,r}^m\,(i\,\gamma)\,P_{n+r}^m\,(\eta) \tag{27}$$

$$S_n^{m\,(j)}\,(-i\,\xi;\,i\,\gamma) = \xi^m\,(\xi^2+1)^{\frac{-m}{2}} \sum_{r \geq m-n} i^r\,a_{n,\,r}^m\,(i\,\gamma)\,\psi_{n+r}^{(i)}\,\frac{\gamma\,\xi}{C_{n\,0}^{(1,\,2)}(i\,\gamma})) \tag{28}$$

where

$$\psi_n^{(1)}\,(k\,r) = \sqrt{\frac{\pi}{2\,k\,r}}\,J_n\,(k\,r), \qquad \psi_n^{(3,\,4)} = \sqrt{\frac{\pi}{2\,k\,r}}\,H_n^{(1,\,2)}\,(k\,r) \tag{29}$$

J_n is the Bessel function, $H_n^{(1,\,2)}$ the Hankel functions of the first or second degree. $C_{n\,0}^m\,(i\,\gamma)$ are defined in (72). The coefficients $a_{n,r}^m\,(i\,\gamma)$ (r = slope) are solutions of a known trinomial recursion system. For the following, it is enough to know the numerical values. Then the spheroid functions are also numerically calculable. Advantageously, we normalize these coefficients such that

$$\sum_{r \geq m+n} \frac{2\,n+1}{2\,n+2\,r+1}\,a_{n,\,1}^m\,(i\,\gamma)\,a_{n,\,r}^{-m}\,(i\,\gamma) = 1 \tag{30}$$

That has the result that the normalization integral for the spherical function P_n^m and the spheroid function Sp_n^m has. The series (27) converges for all finite η, the series (28) for all finite ξ, if $j = 1$, in contrast only for $|\xi| > 1$ if $j = 3$, 4; for $|\xi| < 1$, the value of the function can be calculated through other series representation of the spheroid function. For large ξ

$$S_n^{m\,(1)}(-i\,\xi;\,i\,\gamma) \sim \frac{1}{\gamma\,\xi}\,\cos\left(\gamma\,\xi - \frac{n+1}{2}\,\pi\right) \tag{31}$$

$$S_n^{m\,(3)}(-i\,\xi;\,i\,\gamma) \sim \frac{1}{\gamma\,\xi}\,e^{+i\left(\gamma\,\xi - \frac{n+1}{2}\,\pi\right)} \qquad S_n^{m\,(4)}(-i\,\xi;\,i\,\gamma) \sim \frac{1}{\gamma\,\xi}\,e^{+i\left(\gamma\,\xi - \frac{n+1}{2}\,\pi\right)} \tag{32}$$

Precisely as the spheroid function can be developed in the spherical coordinate system, so can the reverse occur. We obtain (s = degree)

$$P_n^m\,(\eta) = \sum_{s \geq |m|-n}^{\infty} i^s\,a_{n+s,-s}^{-m}\,(i\,\gamma)\,\frac{2\,n+2\,s+1}{2\,n+1} Sp_{n+s}^m\,(\eta;\,i\,\gamma) \tag{33}$$

For those wave functions that are separated in spherical coordinates r, θ, φ we can likewise develop by way of the wave function separated in ξ, $\eta\,\varphi$

$$\psi_n^{(j)}\,(k\,r)\,P_n^m\,(cos\,\vartheta)\,e^{i\,m\,\varphi} = \sum_{s \geq |m|-n}^{\infty} a_{n+s,-s}^{-m}\,(i\,\gamma)\,\frac{2\,n+2\,s+1}{2\,n+1} L_{n+s}^{m(j)}\,(\xi,\,\eta,\,\varphi;\,i\,\gamma) \tag{34}$$

7 Developing the Debye potential of the planar Wave by way of Lamé's Wave Functions

The direction of propagation of the incident planar wave can be assumed, without significant loss of generality, to be parallel to the $x - z$ plane. The wavevector $\mathfrak{k}$ indicates the direction from which the planar wave originates; it includes the angle with the positive $z-$axis. Then

$$\mathfrak{k} = k\,\{\sin\Theta,\, 0,\, \cos\Theta\,\} \qquad \frac{\mathfrak{r}}{r} = \{\,\sin\vartheta\,\cos\varphi,\, \sin\vartheta\,\sin\varphi,\, \cos\vartheta\,\} \tag{35}$$

Now we are to determine the two directions of polarization. We mark the field quantity with the index $\perp$ if the polarization plane of the incident wave is vertical to plane of incidence (here, the $x - z$ plane), and with the index $\|$, if the polarization plane is parallel to the plane of incidence. We expect that the amplitude of the electric vectors of the incident planar wave to E

$$\mathfrak{E}_\perp = \{0,\, E,\, 0\}\; e^{i\,k\,R+i\,\omega\,t}, \qquad \mathfrak{B}_\perp = \sqrt{\frac{\varepsilon}{\mu}}\;\{E\,\cos\Theta,\, 0,\, -E\,sin\Theta\}\; e^{i\,k\,R+i\,\omega\,t}, \tag{36}$$

$$\mathfrak{E}_\| = \{-E\,\cos\Theta,\, 0,\, E\,\sin\Theta\}\; e^{i\,k\,R+i\,\omega\,t}, \qquad \mathfrak{B}_\| = \sqrt{\frac{\varepsilon}{\mu}}\;\{0,\, E,\, 0\}\; e^{i\,k\,R+i\,\omega\,t} \tag{37}$$

where

$$R = r\,(\cos\vartheta\,\cos\Theta + \sin\vartheta\,\sin\Theta\,\cos\varphi) \tag{38}$$

Inserting (36) in (8) provides for the Debye potentials of the incident planar wave in the case that the incident plane is vertical to the direction of oscillation of the differential equations

$$\frac{\partial^2\,(r\,\Pi^i_{1\perp})}{\partial r^2} + k^2\,r\,\Pi^i_{1\perp} = E\,\varepsilon\,\sin\vartheta\,\sin\varphi\cdot e^{i\,k\,R+i\,\omega t} \tag{39a}$$

$$\frac{1}{r}\,\frac{\partial^2(r\,\Pi^i_{1\perp})}{\partial r\partial\vartheta} - i\,k\,\sqrt{\frac{\varepsilon}{\mu}}\,\frac{1}{\sin\vartheta}\,\frac{\partial\Pi^i_{2\perp}}{\partial\varphi} = E\,\varepsilon\,\cos\theta\,\sin\varphi\cdot e^{i\,k\,R+i\,\omega\,t} \tag{39b}$$

$$\frac{1}{r\,\sin\vartheta}\,\frac{\partial^2(r\,\Pi^i_{1\perp})}{\partial r\partial\varphi} + i\,k\,\sqrt{\frac{\varepsilon}{\mu}}\,\frac{\partial\Pi^i_{2\perp}}{\partial\vartheta} = E\,\varepsilon\,\cos\varphi\cdot e^{i\,k\,R+i\,\omega\,t} \tag{39c}$$

The approach of a series separated wave functions in spherical coordinates for both potentials provides, then, a calculation

$$\Pi^i_{1\perp} = -\frac{\varepsilon\,E}{k\,\sin\Theta}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} i^n\,\frac{(2\,n+1)\,m}{n\,(n+1)}(-1)^m\,P_n^{-m}\,(\cos\Theta)\,\psi_n^m\,(r,\,\vartheta,\,\varphi)\,e^{i\,\omega\,t} \tag{40}$$

$$\Pi^i_{2\perp} = +\frac{\sqrt{\varepsilon\,\mu}}{i\,k}\,E\sum_{n=1}^{\infty}\sum_{m=-n}^{n} i^n\,\frac{(2\,n+1)}{n\,(n+1)}(-1)^m\,\frac{dP_n^{-m}}{d\Theta}\,(\cos\Theta)\,\psi_n^m\,(r,\,\vartheta,\,\varphi)\,e^{i\,\omega\,t} \tag{41}$$

Where being substituted for brevity

$$\psi_n^m\,(r,\,\vartheta,\,\varphi) = \psi_n^{(1)}\,(k\,r)\,P_n^m\,(\cos\vartheta)\,e^{i\,m\,\varphi} \tag{42}$$

Corresponding differential equations to (39a, b, c) for $\Pi^i_{1\|}$ and $\Pi^i_{2\|}$ can be found by inserting (37) in (8). From that we obtain

$$\Pi^i_{1\|} = -\sqrt{\frac{\varepsilon}{\mu}}\,\Pi^i_{2\perp}, \qquad \Pi^i_{2\|} = +\sqrt{\frac{\mu}{\varepsilon}}\,\Pi^i_{1\perp} \tag{43}$$

The differential equations (10) are solved by these series for the Debye potentials, as each individual component of the series fulfills the equation.

In order to develop the Debye potentials of the planar wave to get the Lamé wave equations, we develop the wave functions separated in spherical coordinates according to (34) in series after the wave functions separated in ξ, η, φ. Substituted in (40) and (41) yields

$$\Pi^i_{1\perp} = -E\,\frac{\varepsilon}{k}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(2\,n+1)\,m\,i^n\,(-1)^{-m}\,L_n^{m\,(1)}\,(\xi,\,\eta.\,\varphi;\,i\,\gamma)\,U^m_{-n}\,(\Theta)\,e^{i\,\omega\,t} \tag{44}$$

$$\Pi^i_{2\perp} = +E\,\frac{\sqrt{\varepsilon\,\mu}}{i\,k}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(2\,n+1)\,i^n\,(-1)^{-m}\,L_n^{m\,(1)}\,(\xi,\,\eta.\,\varphi;\,i\,\gamma)\,V^m_{n}\,(\Theta)\,e^{i\,\omega\,t} \tag{45}$$

Where being substituted for brevity

$$U^m_n\,(\Theta) = \sum_{r\geq|m|-n}^{\infty}\frac{i^r\,a^{-m}_{n,r}\,(i\,\gamma)}{(n+r)\,(n+r+1)}\,P^{-m}_{n+r}\,(\cos\Theta)\frac{1}{\sin\Theta} \tag{46}$$

$$V^m_n\,(\Theta) = \sum_{r\geq|m|-n}^{\infty}\frac{i^r\,a^{-m}_{n,r}\,(i\,\gamma)}{(n+r)\,(n+r+1)}\,\frac{dP^{-m}_{n+r}\,(\cos\Theta)}{d\Theta} \tag{47}$$

Particularly for $\Theta = 0$, that is to say for a planar wave that runs parallel to the z-axis from positive to negative z, the expressions simplify considerably. Then

$$\Pi^i_{1\perp} = E\,\frac{i\,\varepsilon}{k}\sum_{n=1}^{\infty}\frac{(2\,n+1)\,i^n}{n\,(n+1)}\,C^1_{n\,0}\,(i\,\gamma)\,S_n^{1\,(1)}\,(-i\,\xi;\,i\,\gamma)\,Sp^1_n\,(\eta;\,i\,\gamma)\,\sin\varphi\cdot e^{i\,\omega\,t} \tag{48}$$

$$\Pi^i_{2\perp} = E\,i\,\frac{\sqrt{\varepsilon\,\mu}}{k}\sum_{n=1}^{\infty}\frac{(2\,n+1)\,i^n}{n\,(n+1)}\,C^1_{n\,0}\,(i\,\gamma)\,S_n^{1\,(1)}\,(-i\,\xi;\,i\,\gamma)\,Sp^1_n\,(\eta;\,i\,\gamma)\,\cos\varphi\cdot e^{i\,\omega\,t} \tag{49}$$

This series development converges for all space.

8 Calculation of the first Part of the diffracting Wave

We decompose the Debye potentials into two parts

$$\Pi^r_1 = \overline{\Pi}^r_1 + \overline{\overline{\Pi}}^r_1, \qquad \Pi^r_2 = \overline{\Pi}^r_2 + \overline{\overline{\Pi}}^r_2 \tag{50}$$

We determine the first part from the claim

$$\Pi^i_1 + \overline{\Pi}^r_1 = 0, \qquad \frac{\partial}{\partial z}\,(\Pi^i_2 + \Pi^r_2) = 0, \qquad \text{for } \xi = 0 \tag{51}$$

Then, for the second part, it follows by (17)

$$\varrho\,\overline{\overline{\Pi}}^r_1 = \alpha\,(\varphi)\,e^{i\,k\,\varrho} + \beta\,(\varphi)\,e^{-i\,k\,\varrho}; \qquad \sqrt{\frac{\varepsilon}{\mu}}\,\varrho^2\,\frac{\partial\overline{\overline{\Pi}}^r_2}{\partial z} = \frac{d\alpha}{d\varphi}\,e^{i\,k\,\varrho} - \frac{d\beta}{d\varphi}\,e^{-i\,k\,\varrho} \tag{52}$$

Both $\overline{\Pi}^r_1$, $\overline{\Pi}^r_2$ and $\overline{\overline{\Pi}}^r_1$, $\overline{\overline{\Pi}}^r_2$ depict outgoing waves and can be assembled additively from Lamé's wave functions $L_n^{m\ (4)}\,(\xi,\,\eta,\,\varphi;\,i\,\gamma)$. It can be easily proven that the following statements fulfill (51) for $\overline{\Pi}^r_1$, $\overline{\Pi}^r_2$.

$$\overline{\Pi}^r_{1\perp} \quad = +E\,\frac{\varepsilon}{k}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(2\,n+1)\,i^n\,m\,(-1)^m\,L_n^{m\,(4)}\,(\xi,\,\eta,\,\varphi;\,i\,\gamma)\,U^m_n\,(\Theta)\,\frac{S_n^{m\,(1)}\,(-i\,0;\,i\,\gamma)}{S_n^{m\,(4)}\,(-i\,0;\,i\,\gamma)}\,e^{i\,\omega\,t} \tag{53}$$

$$\overline{\Pi}^r_{2\perp} = -E\,\frac{\sqrt{\varepsilon\,\mu}}{i\,k}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(2\,n+1)\,i^n\,(-1)^m\,L_n^{m\,(4)}\,(\xi,\,\eta,\,\varphi;\,i\,\gamma)\,V_n^m\,(\Theta)\,\frac{dS_n^{m\,(1)}\,(-i\,0;\,i\,\gamma)/d\xi}{dS_n^{m\,(4)}\,(-i\,0;\,i\,\gamma)/d\xi}\,e^{i\,\omega\,t} \quad (54)$$

$$\overline{\Pi}^r_{1\parallel} = +E\,\frac{\varepsilon}{i\,k}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(2\,n+1)\,i^n\,(-1)^m\,L_n^{m\,(4)}\,(\xi,\,\eta,\,\varphi;\,i\,\gamma)\,V_n^m\,(\Theta)\,\frac{S_n^{m\,(1)}\,(-i\,0;\,i\,\gamma)}{S_n^{m\,(4)}\,(-i\,0;\,i\,\gamma)}\,e^{i\,\omega\,t} \quad (55)$$

$$\overline{\Pi}^r_{2\parallel} = +E\,\frac{\sqrt{\varepsilon\,\mu}}{i\,k}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(2\,n+1)\,i^n\,m\,(-1)^m\,L_n^{m\,(4)}\,(\xi,\,\eta,\,\varphi;\,i\,\gamma)\,U_n^m\,(\Theta)\,\frac{dS_n^{m\,(1)}\,(-i\,0;\,i\,\gamma)/d\xi}{dS_n^{m\,(4)}\,(-i\,0;\,i\,\gamma)/d\xi}\,e^{i\,\omega\,t} \quad (56)$$

It is remarkable that the components with odd differences $n-m$ in (53) and (55) and those with even $n-m$ in (54) and (56) disappear due to the behavior of the first kind of spheroid functions and their derivative for the zero argument. These four written functions satisfy all of the wave functions, behave at large distance as outgoing spherical waves and are at all places finite.

for $\Theta = 0$, there are again considerable simplifications; they are

$$\overline{\Pi}^r_{1\perp} = -E\,\frac{i\,\varepsilon}{k}\sum_{n=1}^{\infty}\frac{2\,n+1}{n\,(n+1)}\,i^n\,C^1_{n\,0}\,(i\,\gamma)\,S_n^{1\,(4)}(-i\,\xi;\,i\,\gamma)\,Sp_n^{1\,(1)}\,(\eta;\,i\,\gamma)\,\sin\varphi\,\frac{S_n^{1\,(1)}\,(-i\,0;\,i\,\gamma)}{S_n^{1\,(4)}\,(-i\,0;\,i\,\gamma)}\,e^{i\,\omega\,t} \quad (57)$$

$$\overline{\Pi}^r_{2\perp} = -E\,i\,\frac{\sqrt{\varepsilon\,\mu}}{k}\sum_{n=1}^{\infty}\frac{2\,n+1}{n\,(n+1)}\,i^n\,C^1_{n\,0}\,(i\,\gamma)\,S_n^{1\,(4)}(-i\,\xi;\,i\,\gamma)\,Sp_n^{1\,(1)}\,(\eta;\,i\,\gamma)\,\cos\varphi\,\frac{dS_n^{1\,(1)}\,(-i\,0;\,i\,\gamma)/d\xi}{dS_n^{1\,(4)}\,(-i\,0;\,i\,\gamma)/d\xi}\,e^{i\,\omega\,t} \quad (58)$$

These series also converge for all space. In the case $\Theta = 0$, a separate calculation of $\overline{\Pi}^r_{1\parallel}$ and $\overline{\Pi}^r_{2\parallel}$ is unnecessary, as the cases of parallel and perpendicular polarization planes differ in field strength only by a phase of 90°.

9 Calculation of the second Part of the Diffracting Wave

We think first of the still unknown functions $\alpha\,(\varphi)$ and $\beta\,(\varphi)$ in (52) as a Fourier series

$$\alpha\,(\varphi) = \sum_{m=-\infty}^{\infty}\alpha_m\,e^{i\,m\,\varphi}, \qquad \beta\,(\varphi) = \sum_{m=-\infty}^{\infty}\beta_m\,e^{i\,m\,\varphi} \quad (59)$$

For $\overline{\overline{\Pi}}^r_1$ and $\overline{\overline{\Pi}}^r_2$, the wave functions must describe to be outgoing spherical waves at large distance, and fulfill the boundary constraints (52) for $\xi = 0$. We say

$$\overline{\overline{\Pi}}^r_1 = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_n^m\,S_n^{;(4)}\,(-i\,\xi;\,,i\,\gamma)\,Sp_n^m\,(\eta;\,i\,\gamma)\,e^{i\,m\,\varphi} \quad (60)$$

$$\overline{\overline{\Pi}}^r_2 = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}B_n^m\,S_n^{;(4)}\,(-i\,\xi;\,,i\,\gamma)\,Sp_n^m\,(\eta;\,i\,\gamma)\,e^{i\,m\,\varphi} \quad (61)$$

The boundary constraints (52) now provide, under consideration from (59)

$$\sum_{n=0}^{\infty}A_n^m\,S_n^{(4)}\,(-i\,0;\,i\,\gamma)\,Sp_n^m\,(\eta;\,i\,\gamma) = \alpha_m\,\frac{e^{i\,\gamma\,\sqrt{1-\eta^2}}}{c\,\sqrt{1-\eta^2}} + \beta_m\,\frac{e^{-i\,\sqrt{1-\eta^2}}}{c\,\sqrt{1-\eta^2}} \quad (62)$$

$$\sum_{n=0}^{\infty} B_n^m \left[\frac{d}{d\xi}\left(S_n^{m\,(4)}\,(-i\,\xi;\,i\,\gamma)\right)\right]_{\xi=0} Sp_n^m\,(\eta;\,i\,\gamma) = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{i\,m\,\eta}{c\,(1-\eta^2)}\left[\alpha_m\,e^{i\,\gamma\,\sqrt{1-\eta^2}} - \beta_m\,e^{-i\,\gamma\,\sqrt{1-\eta^2}}\right] \tag{63}$$

It is immediately noticeable, that

$$A_n^m = 0 \text{ for } n-m = \text{odd} \qquad B_n^m = 0 \text{ for } n-m = \text{even} \tag{64}$$

The orthogonality and normalization terms

$$\int_{-1}^{1} Sp_n^m\,(\eta;\,i\,\gamma)\,Sp_l^{-m}\,(\eta;\,i\,\gamma)\,d\eta = \begin{cases} 0 & \text{for } n \neq l \\ \frac{2}{2\,n+1} & \text{for } n = l \end{cases} \tag{65}$$

allow the coefficients A_n^m and B_n^m to be expressed by α_m and β_m. The conditions

$$\frac{\partial}{\partial\eta}\left(\Pi_1^i + \overline{\Pi}_1^r + \overline{\overline{\Pi}}_1^r\right) = 0 \quad \text{and} \quad \frac{\partial}{\partial\xi}\left(\Pi_2^i + \overline{\Pi}_2^r + \overline{\overline{\Pi}}_2^r\right) = 0 \quad \text{for } \xi = \eta = 0 \tag{66}$$

from (21) are fulfilled; because by (15) and (16), for $\eta = 0$, that is to say on the disk, Π_1 and $\frac{\partial\Pi_2}{\partial z} = \frac{1}{c\,\eta}\,\frac{\partial\Pi_2}{\partial\xi}$ are even functions in η; (66) hence applies for all η, if $\xi = 0$. The conditions

$$\frac{\partial}{\partial\xi}\left(\Pi_1^i + \overline{\Pi}_1^r + \overline{\overline{\Pi}}_1^r\right) = 0 \quad \text{and} \quad \frac{\partial}{\partial\eta}\left(\Pi_2^i + \overline{\Pi}_2^r + \overline{\overline{\Pi}}_2^r\right) = 0 \quad \text{for } \xi = \eta = 0 \tag{67}$$

from (21) give just two equations for each pair of coefficients α_m, β_m. Thereby, the solution for the diffraction problem is thoroughly certain.

The ratio of the incident wave may be omitted in (67), because the derivative with respect to ξ and η disappears on the screen edge. Physically, this follows from the the finiteness of the incident wave, since there can be no infinite field energy on the edge of the screen.

10 Calculation of the series Coefficients A_n^m and B_n^m

The orthogonality and normalization terms (65) give

$$A_n^m\,S_n^{m\,(4)}\,(-i\,0;\,i\,\gamma)\,\frac{2}{2\,n+1} = \frac{1}{c}\int_{-1}^{1}\left[\alpha_m\,\frac{e^{i\,\gamma\,\sqrt{1-\eta^2}}}{\sqrt{1-\eta^2}} + \beta_m\,\frac{e^{-i\,\gamma\,\sqrt{1-\eta^2}}}{\sqrt{1-\eta^2}}\right] Sp_n^{-m}\,(\eta;\,i\,\gamma)\,d\eta \tag{68}$$

$$B_n^m\left[\frac{d}{d\xi}\,S_n^{m\,(4)}\,(-i\,\xi;\,i\,\gamma)\right]_{\xi=0}\frac{2}{2\,n+1} = \frac{i\,m}{c}\sqrt{\frac{\mu}{\varepsilon}}\int_{-1}^{1}\frac{\eta}{1-\eta^2}\left[\alpha_m\,e^{i\,\gamma\,\sqrt{1-\eta^2}} - \beta_m\,e^{-i\,\gamma\,\sqrt{1-\eta^2}}\right] Sp_n^{-m}\,(\eta;\,i\,\gamma)\,d\eta \tag{69}$$

These integrals can be evaluated as closed integrals. However, we do without the calculation, which should be successful in other contexts, and satisfies us with the statement of its value for $m = +1$. We only need these, if we want to be confined, as occurs in the following for the case $\Theta = 0$, that is to say perpendicular incidence to the planar wave on the disk. Then

$$\begin{aligned} A_n^1\,S_n^{1\,(4)}\,(-i\,0;\,i\,\gamma)\,\frac{n\,(n+1)}{2\,n+1}\,c = {} & \left[+C_{n\,0}^1\,(i\,\gamma) - \gamma\,\alpha_{n,-n-1}^1\,(i\,\gamma)\,S_n^{1\,(3)}\,(-i\,0;\,i\,\gamma)\right]\alpha_1 \\ & + \left[+C_{n\,0}^1\,(i\,\gamma) - \gamma\,\alpha_{n,-n-1}^1\,(i\,\gamma)\,S_n^{1\,(4)}\,(-i\,0;\,i\,\gamma)\right]\beta_1; \\ & (n = \text{odd}) \end{aligned} \tag{70}$$

$$B_n^1 \Big[\frac{d}{d\xi}\, S_n^{1\,(4)}\,(-i\,\xi;\, i\,\gamma)\Big]_{\xi=0}\, \frac{n\,(n+1)}{2\,n+1}\, c\sqrt{\frac{\varepsilon}{\mu}} = \gamma\,\Big[-C_{n\,0}^1\,(i\,\gamma) + \alpha_{n,-n}^1\,(i\,\gamma)\,\big[\frac{d}{d\xi}\, S_n^{1\,(3)}\,(-i\,\xi;\, i\,\gamma)\big]_{\xi=0}\Big]\,\alpha_1 + \gamma\,\Big[-C_{n\,0}^1\,(i\,\gamma) + \alpha_{n,-n}^1\,(i\,\gamma)\,\big[\frac{d}{d\xi}\, S_n^{1\,(3)}\,(-i\,\xi;\, i\,\gamma)\big]_{\xi=0}\Big]\,\beta_1 \quad (n = \text{even}) \tag{71}$$

For brevity, we substitute

$$C_{n\,0}^1\,(i\,\gamma) = \sum_{r\geq -n-1}^{\infty} i^n\,\alpha_{n,\,r}^1\,(i\,\gamma) \tag{72}$$

As $\Pi_1^i,\ \overline{\Pi}_1^i$ remain proportional to $\sin\varphi$ and $\Pi_2^i,\ \overline{\Pi}_2^i$ to $\cos\varphi$, then by (67) the same relations must hold for $\overline{\overline{\Pi}}_1^i$ and $\overline{\overline{\Pi}}_2^i$. Therefore it suffices to calculate $A_n^1,\ B_n^1$ and to multiply the corresponding series components in (60) and (61) by $e^{i\,\varphi} \mp e^{i\,\varphi}$.

Hence, for the overall result, we get

$$\Pi_1 = E\,\frac{i\,\varepsilon}{k}\sum_{n=1}^{\infty}\frac{2\,n+1}{n\,(n+1)}\, i^n\, C_{n\,0}^1\,(i\,\gamma)\left[S_n^{1\,(1)}\,(-i\,\xi;\, i\,\gamma) - S_n^{1\,(4)}\,(-i\,\xi;\, i\gamma)\,\frac{S_n^{1\,(1)}\,(-i\,0;\, i\,\gamma)}{S_n^{1\,(4)}\,(-i\,0;\, i\,\gamma)}\right] Sp_n^1\,(\eta;\, i\,\gamma) \cdot \sin\varphi\, e^{i\,\omega\,t} + 2\,i\,\sin\varphi\sum_{n=1}^{\infty} A_n^1\, S_n^{1\,(4)}\,(-i\,\xi;\, i\,\gamma)\, Sp_n^1\,(\eta;\, i\,\gamma)\, e^{i\,\omega\,t} \tag{73}$$

$$\Pi_2 = E\,\frac{i\,\sqrt{\varepsilon\,\mu}}{k}\sum_{n=2}^{\infty}\frac{2\,n+1}{n\,(n+1)}\, i^n\, C_{n\,0}^1\,(i\,\gamma)\left\{S_n^{1\,(1)}\,(-i\,\xi;\, i\,\gamma) - S_n^{1\,(4)}\,(-i\,\xi;\, i\gamma)\left[\frac{dS_n^{1\,(1)}\,(-i\,0;\, i\,\gamma)/d\xi}{dS_n^{1\,(4)}\,(-i\,0;\, i\,\gamma)/d\xi}\right]_{\xi=0}\right\} \cdot Sp_n^1\,(\eta;\, i\,\gamma)\cos\varphi\, e^{i\,\omega\,t} + 2\,\cos\varphi\sum_{n=2}^{\infty} B_n^1\, S_n^{1\,(4)}\,(-i\,\xi;\, i\,\gamma)\, Sp_n^1\,(\eta;\, i\,\gamma)\, e^{i\,\omega\,t} \tag{74}$$

The hitherto unknown quantities α_1 and β_1 yield that $\frac{\partial\Pi_1}{\partial\xi} = 0$ and $\frac{\partial\Pi_2}{\partial\eta} = 0$ for $\xi = \eta = 0$. This yields two linear equations for α_1 and β_1.

11 Borderline Case of long Wavelengths

If the wavelength is large compared to the radius of the disk, that is to say $\gamma \ll 1$, then the spheroid functions become expressible in spherical functions of the first or second kind if η and ξ remain finite. From the magnitudes $S_n^{1\,(1)}\,(-i\,\xi;\, i\,\gamma) = \mathcal{O}\,(\gamma^r)$, $S_n^{1\,(3,\,4)}\,(-i\,\xi;\, i\,\gamma) = \mathcal{O}\,(\gamma^{-n-1})$ follows that in (73) and (74) only the first component remains from the respective first series. For the second, all components are to be considered. The differential quotient of this second series in terms of ξ or η, which is required to utilize the edge conditions (21), is thus developed to be a divergent series (alternating series with the marginal component of the order $n^{1/2}$. These difficulties can be avoided, however, by indeed differentiating term by term, and then by a summation method (about the arithmetic mean), the sum calculates. This is feasibly closed for long wavelengths and yields

$$\alpha_1 + \beta_1 = E\,\frac{\varepsilon\, i\,\gamma^2}{2\,k^2}\Big(1 + \mathcal{O}\,(\gamma)\Big) \qquad \alpha_1 - \beta_1 = -E\,\frac{\varepsilon\,\gamma^3}{9\,\pi\,k^2}\Big(1 + \mathcal{O}\,(\gamma)\Big) \tag{75}$$

With these values for α_1 and β_1 we can calculate the entire field. In the wave area where $\gamma\,\xi \gg 1$, as the asymptotic representations (31) and (32) in addition to the above show, from all series in (73) and (74) only the first component remain respectively, and this gives as $\gamma\,\xi \approx k\,r$ for Π_1^r

$$\Pi_1^r = E\,\frac{4\,\gamma^3\,\varepsilon}{3\,\pi\, i\, k}\,\frac{e^{-i\,k\,r}}{r}\, P_1^1\,(\cos\varphi)\,\sin\varphi\, e^{i\,\omega t} \tag{76}$$

The corresponding radiation (76) is that of an electric dipole with the moment

$$p_y = E\,\frac{16}{3}\,\alpha^3\,\varepsilon\,e^{i\,\omega\,t} \tag{77}$$

This is the dipole moment of the charge density that would be induced in a disk if it finds itself in a uniform electric field $E\,e^{i\,\omega\,t}$ (for $\gamma \ll$, that is to say quasi-statistically calculated) parallel to the $y-$ axis.

We forgo the calculation and interpretation of of Π_1^r, as it already is the first member of the series as was neglected in Π_1^r.

12 Remarks about the numerical Evaulation

The numerical evaluation of the formulas (73) and (74) assumes the existence of panels of spheroid functions. Such were calculated by Stratton and Mitarbb[8] but they have not yet the desired range. The same goes for the ones by the author[9] for the developments of $\alpha_{n,r}^m\,(i\,\gamma)$ for powers of γ, which for small n is necessary only until $\gamma = 1$. The area $0 \leq \gamma \leq 10$ and $1 \leq n \leq 12$ would be important. For such values of γ the number to be considered the series component (namely, approximately until $n = 12$) is bearable, moreover a good connection to Kirchhoff's approximation with $\gamma = 10$, as is also the case for the scalar diffraction problem. The numerical calculation of the required spheroidal functions creates few difficulties of either theoretical or practical nature.

One difficult point is the already mentioned divergence of the series, which is given when we differentiate (73) and (74) with respect to ξ or η and set them equal to zero. Because the application of a summation technique, this consideration requires many series components. A loophole, however, can be found in the fact that the series components for larger n converge rapidly compared with the equivalent series components for the case $\gamma = 0$, for which the sum of the divergent series can be closed and calculated. Therefore the corresponding series for $\gamma = 0$ will be subtracted member-by-member from the series to be summed for $\gamma \neq 0$. Then develops convergent series, whose summation is fundamentally easier.

The still-to-undertaken numerical evaluation of the field of the diffracting wave, which we want to describe in a later work, will be reached mainly for the field at large distance and in the neighborhood of the disk or the circular opening, with the goal to obtain information about the polarization ratios. The results gain interest through newer experimental research of the field in the vicinity of circular opening by wavelengths in the order of magnitude of the opening, which were conducted by Severin[10] and Andrews[11] with wavelengths from 6 until 12.8 cm.

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