How to beat your friends at the dots-and-boxes game!

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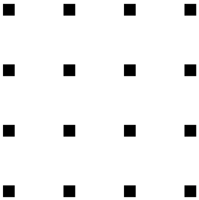
Not: Backgammon (has dice), Poker (has card deal), Stratego (has hidden information), Monopoly (has dice and >2 players sometimes)

Dots-and-boxes - how to play

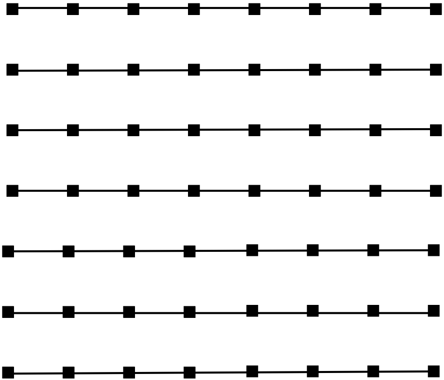
- Played on a rectangular grid with m rows, each containing n dots.
- A move consists of drawing a horizontal or vertical line connecting two dots.
- Upon completing a box, a player 'claims' it and moves again.
- Play ends when all possible lines are drawn.
- The winner is the person who has claimed the most boxes. If both players have the same number of boxes, then it is a tie.

Let's play!

The 9-box game (4×4-game).



Let's play from here (49 moves and turns played) - player A plays odd numbered turns, player B plays even turns:



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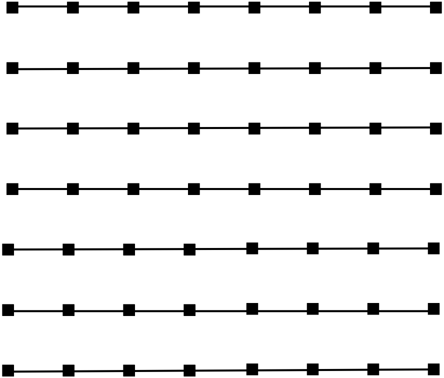

What is the **greedy** approach?

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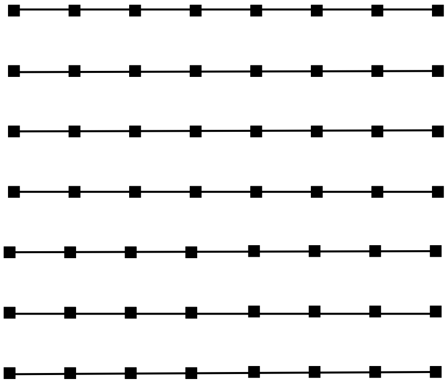
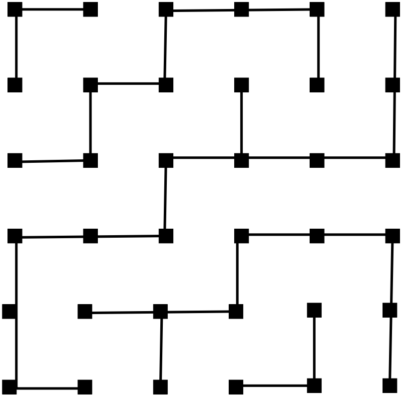
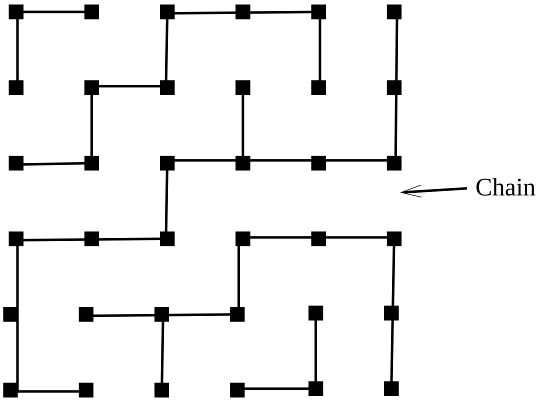
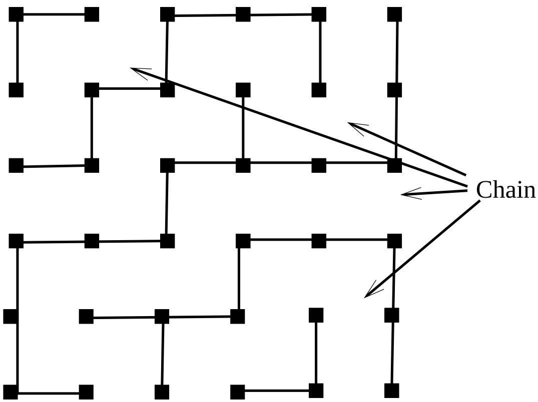


Don't take the **greedy** approach! Let's make *double-dealing* moves. Your opponent will be *double-crossed* (forced to take two boxes with one pen-stroke).




Chain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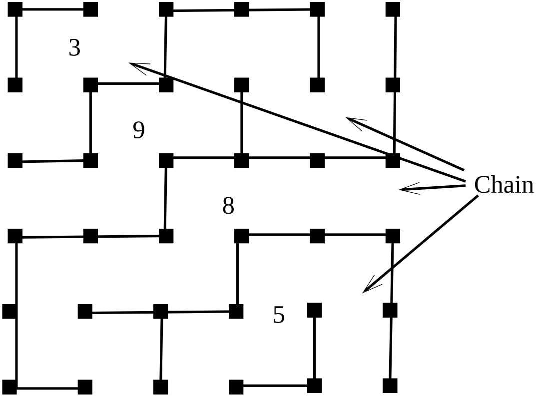
3
9
8
5
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- Once you have control, KEEP CONTROL, dude, by (politely) declining 2 boxes of every long chain except the last. (So, you will be last to play.)

To get control

Player A tries to make # of initial dots + #double-crossed moves odd.
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WHY? # of initial dots + # of double-crosses = total # of turns in game.

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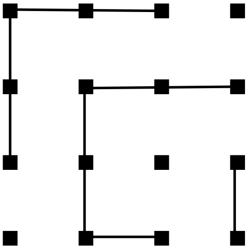
Chain Counting Problems - Puzzle 1



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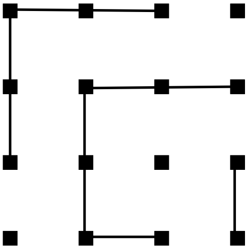

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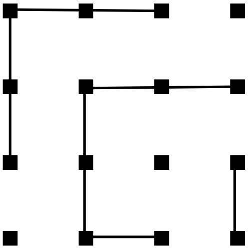

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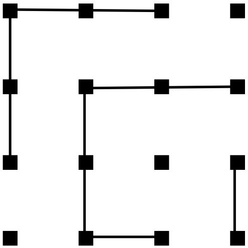
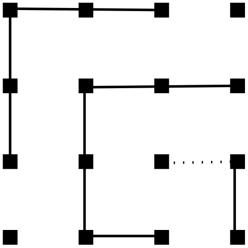


10 moves made, so it is A's turn. # dots is 16, an even number. By the long chain rule, A wants an even number of long chains.

Chain Counting Problems - Solution 1



The dashed line indicates best move. It ensures 2 chains.

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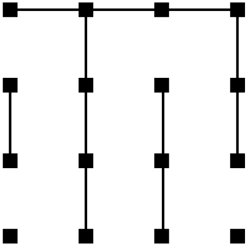

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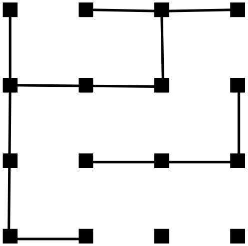

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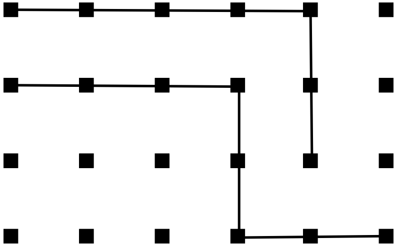

Thanks!!

- Talk based on book by Elwyn Berlekamp entitled "The Dots-and-Boxes Game: Sophisticated Child's Play."
- Slides available from my homepage: http://community.middlebury.edu/~jschmitt/.