LINEAR ALGEBRA
EXAM 3
FALL 2024

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Signature:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Good luck!

average
53 of 60
- wow!

(1) [5 points] Suppose that $\mathbf{x}$ is an eigenvector of the matrix B with eigenvalue 3. For this to be true, write the equation that must be satisfied. Then multiply both sides of this equation by B. Deduce what an eigenvalue for B^2 must be.

Then $\vec{x}$ must be a nonzero vector satisfying

$$B\vec{x} = 3\vec{x}$$

If we multiply both sides by B we obtain

$$B^2\vec{x} = B3\vec{x} = 3B\vec{x} = 3\,3\vec{x} = 9\vec{x}$$

So 9 is an eigenvalue of B^2.

Date: December 11, 2024.

(2) [10 points] Show that $\lambda = 1$ is an eigenvalue of the following matrix. Find a basis for the eigenspace. State the dimension of the eigenspace.

$$A= \begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$$

To show that 1 is an eigenvalue of A, we must find a non-trivial solution to $A\vec{x} = 1\vec{x}$. We use G.E. on $[A - 1I / \vec{0}] = \left[\begin{array}{ccc|c} 3 & -2 & 3 & 0 \\ 0 & -2 & 3 & 0 \\ -1 & 2 & -3 & 0 \end{array}\right]$

This is row equivalent to

$$\left[\begin{array}{ccc|c} 3 & -2 & 3 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 4/3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & -2 & 3 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{3}R1 + R3$ $\qquad$ $\frac{2}{3}R2 + R3$ $\qquad$ $-R2 + R1$

So, x_3 is a free variable, $x_1 = 0$ and $x_2 = \frac{3}{2}x_3$.

Thus, the eigenspace has dimension 1 and

$\vec{x} = x_3 \begin{bmatrix} 0 \\ 3/2 \\ 1 \end{bmatrix}$ is the solution set.

A basis for the eigenspace is $\left\{ \begin{bmatrix} 0 \\ 3/2 \\ 1 \end{bmatrix} \right\}$.

(3) [5 points] Give an example of a 2×2 matrix that is not triangular for which 0 is not an eigenvalue. Justify that 0 is not an eigenvalue.

The Invertible Matrix Theorem says that matrix invertibility and 0 is not an eigenvalue are equivalent. The following matrix is not triangular and is invertible:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{since} \quad \det(A) = 1 \neq 0.$$

(4) [5 points] Find the characteristic polynomial and the real eigenvalues of the following matrix.

$$P = \begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$$

The characteristic polynomial is

$$\det(P - \lambda I) = \det\left(\begin{bmatrix} 9-\lambda & -2 \\ 2 & 5-\lambda \end{bmatrix}\right)$$

$$= (9-\lambda)(5-\lambda) - (-4)$$

$$= \lambda^2 - 14\lambda + 49$$

Setting this to zero and solving for λ finds us the eigenvalues.

$$\lambda^2 - 14\lambda + 49 = 0$$

$$(\lambda - 7)^2 = 0$$

$\lambda = 7$ with multiplicity 2.

(5) [10 points] Consider the following matrix C, which has eigenvalues 5 and -2. The eigenspace corresponding to eigenvalue 5 has the vector $\mathbf{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ as a basis vector. The eigenspace corresponding to eigenvalue -2 has the vector $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as a basis vector. Use this information to compute C^4. (One may leave the entries in C^4 in factored form.)

$$C = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

Using the given information, we may find a diagonalization of C as PDP^{-1} where $P = \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$. We may find P^{-1} thru G.E. on

$$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right] \sim \underset{-\frac{4}{3}R1 + R2}{\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 0 & 7/3 & -4/3 & 1 \end{array}\right]} \sim \underset{\frac{3}{7}R2}{\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \end{array}\right]}$$

$$\sim \underset{\frac{1}{3}R1}{\left[\begin{array}{cc|cc} 1 & -1/3 & 1/3 & 0 \\ 0 & 1 & -4/7 & 3/7 \end{array}\right]} \sim \underset{\frac{1}{3}R2 + R1}{\left[\begin{array}{cc|cc} 1 & 0 & 1/7 & 1/7 \\ 0 & 1 & -4/7 & 3/7 \end{array}\right]}$$

So $P^{-1} = \begin{bmatrix} 1/7 & 1/7 \\ -4/7 & 3/7 \end{bmatrix}$. Thus, $C^4 = PD^4P^{-1}$

$$= \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 625 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1/7 & 1/7 \\ -4/7 & 3/7 \end{bmatrix}$$

(6) [5 points] Let $\mathbf{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through the origin and $\mathbf{u}$.

First we compute the projection of $\vec{y}$ onto the span of $\vec{u}$.

$$\hat{y} = \text{proj}_{\vec{u}}\, \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\, \vec{u} = \frac{-3+18}{1+4} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{y} = \frac{15}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

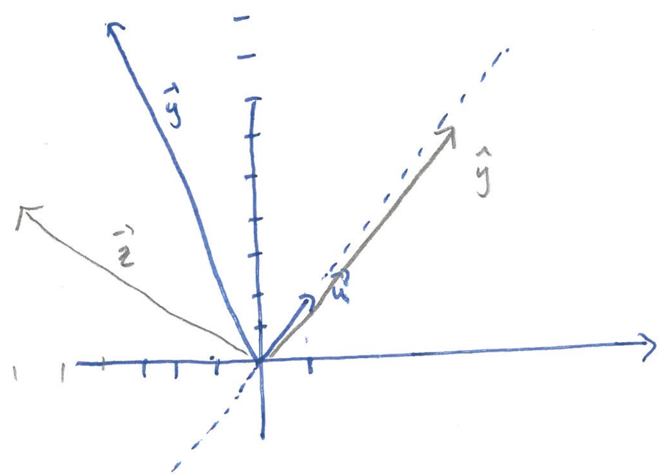
Then the orthogonal component of $\vec{y}$ is

$$\vec{y} - \hat{y} = \vec{z} \implies \begin{bmatrix} -3 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}.$$

The length of $\vec{z}$ is the distance we seek.

$$\|\vec{z}\| = \sqrt{(-6)^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$$

Here is the picture



(7) [10 points] Find the orthogonal projection of **b** onto Col A and a least-squares solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

We can use the Orthogonal Decomposition Theorem if we <u>first</u> show that the columns of A form an orthogonal set - this allows us to find $\hat{x}$ and proj $\vec{b}$ simultaneously.

$$\vec{a}_1 \cdot \vec{a}_2 = 1 + 0 + 0 - 1 = 0$$
$$\vec{a}_1 \cdot \vec{a}_3 = 0 - 1 + 0 + 1 = 0$$
$$\vec{a}_2 \cdot \vec{a}_3 = 0 + 0 + 1 - 1 = 0.$$

Thus,
$$\text{proj}_{\text{Col}A}\vec{b} = \hat{b} = \frac{\vec{b}\cdot\vec{a}_1}{\vec{a}_1\cdot\vec{a}_1}\vec{a}_1 + \frac{\vec{b}\cdot\vec{a}_2}{\vec{a}_2\cdot\vec{a}_2}\vec{a}_2 + \frac{\vec{b}\cdot\vec{a}_3}{\vec{a}_3\cdot\vec{a}_3}\vec{a}_3$$
$$= \frac{1}{3}\vec{a}_1 + \frac{14}{3}\vec{a}_2 + \frac{(-5)}{3}\vec{a}_3$$
$$= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{14}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

The weights are the entries in $\hat{x}$.

An alternate solution is to solve the normal equations
$$A^T A \vec{x} = A^T \vec{b}.$$

(7) [10 points] Find the orthogonal projection of **b** onto Col A and a least-squares solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

We may solve the normal equations as follows.

$A^T A \hat{x} = A^T \vec{b}$ — this finds $\hat{x}$.

$$A^T A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$$

G.E. on $\left[\begin{array}{ccc|c} 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 14 \\ 0 & 0 & 3 & -5 \end{array}\right]$

yields $\hat{x} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$

(6) [5 points] Let $\mathbf{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through the origin and $\mathbf{u}$.

(8) [10 points] **Google's PageRank Algorithm** Construct a web (drawing it below) on six nodes for which the dimension of the eigenspace of the link matrix A corresponding to the eigenvalue 1 is 2-dimensional **and** for which the number of non-zero entries of A is as small as possible. Then give the modified link matrix M with the choice of m as $1/10$ while also stating the equation that one must solve to find the importance scores of each webpage. (You needn't solve this equation.)

For the eigenspace to be 2-dimensional, the # of components (i.e. disconnected subwebs) must be 2. For the # of non-zero entries of A to be as small as possible, the number of links should be as small as possible. The following web achieves these two criteria: (and avoids dangling nodes)

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Then let $S = \begin{bmatrix} 1/6 & \cdots & 1/6 \\ \vdots & \ddots & \\ 1/6 & & 1/6 \end{bmatrix}$

Then $M = (1 - 1/10)\, A + \frac{1}{10} S$.

One needs to solve $M\vec{x} = \vec{x}$ to find the importance scores.