LINEAR ALGEBRA
EXAM 3
FALL 2024

Name:
Honor Code Statement:

Signature:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Good luck!

(1) [5 points] Suppose that $\mathbf{x}$ is an eigenvector of the matrix B with eigenvalue 3. For this to be true, write the equation that must be satisfied. Then multiply both sides of this equation by B. Deduce what an eigenvalue for B^2 must be.

Date: December 11, 2024.

(2) [10 points] Show that $\lambda = 1$ is an eigenvalue of the following matrix. Find a basis for the eigenspace. State the dimension of the eigenspace.

$$A= \begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$$

(3) [5 points] Give an example of a 2×2 matrix that is not triangular for which 0 is not an eigenvalue. Justify that 0 is not an eigenvalue.

(4) [5 points] Find the characteristic polynomial and the real eigenvalues of the following matrix.

$$P = \begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$$

(5) [10 points] Consider the following matrix C, which has eigenvalues 5 and -2. The eigenspace corresponding to eigenvalue 5 has the vector $\mathbf{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ as a basis vector. The eigenspace corresponding to eigenvalue -2 has the vector $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as a basis vector. Use this information to compute C^4. (One may leave the entries in C^4 in factored form.)

$$C = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

(6) [5 points] Let $\mathbf{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through the origin and $\mathbf{u}$.

(7) [10 points] Find the orthogonal projection of $\mathbf{b}$ onto Col A and a least-squares solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

(8) [10 points] **Google's PageRank Algorithm** Construct a web (drawing it below) on six nodes for which the dimension of the eigenspace of the link matrix A corresponding to the eigenvalue 1 is 2-dimensional **and** for which the number of non-zero entries of A is as small as possible. Then give the modified link matrix M with the choice of m as $1/10$ while also stating the equation that one must solve to find the importance scores of each webpage. (You needn't solve this equation.)