LINEAR ALGEBRA
EXAM 3
FALL 2023

Name:
Honor Code Statement:

Signature:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Good luck!

(1) [10 points] For the given matrix A confirm that $\lambda = 3$ is an eigenvalue for A by finding a basis for the eigenspace corresponding to this eigenvalue.

$$A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}$$

Date: December 14, 2023.

(2) The following statements are false. Give a counter-example to each. That is, give an instance where the hypothesis is true but the conclusion is false. Justify your counter-example.

(a) [5 points] The eigenvalues of a 2×2 matrix A are the entries on the main diagonal.

(b) [5 points] Every linearly independent set of size 2 in $\mathbb{R}^3$ is an orthogonal set.

(3) [10 points] Begin by finding the characteristic equation of the following matrix.

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Next, confirm that $\lambda = 2$ and $\lambda = 3$ are eigenvalues of this matrix A by plugging these values into the equation you just obtained.

Now find a diagonalization of the matrix A.

(4) [5 points] Suppose that $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to every $\mathbf{w}$ in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$.

(5) [5 points] Find the distance between the two vectors $\mathbf{a} = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(6) [10 points] Confirm that the given set is an orthogonal set.

$$S=\{\ \mathbf{u_1} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \}$$

What theorem confirms that this set is a basis for $\mathbb{R}^3$?

Now write the following vector $\mathbf{x}$ as a linear combination of the vectors in S *without* doing Gaussian Elimination.

$$\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

(7) [5 points] Apply the Gram-Schmidt process to produce an orthogonal basis for the subspace spanned by the following set of vectors.

$$S=\{\ \mathbf{a_1}=\begin{bmatrix} 4 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \mathbf{a_2}=\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}\}$$

(8) [10 points] Find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ where $A = [\mathbf{a_1}\ \mathbf{a_2}]$, with $\mathbf{a_1}$ and $\mathbf{a_2}$ coming from the previous problem and $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}$.

(9) [10 points] Consider the small web below. Write down the link matrix A for this web. (Note that I'm asking for the link matrix A, not the modified link matrix M.) This link matrix will *not* yield a unique ranking. State what computation could be done to demonstrate this and what outcome of that computation must be found to hold true. Notice that I'm not asking for a qualitative argument about the web (i.e., it's disconnected); I'm asking for a computational way to demonstrate that the ranking is not unique. Please do not do the computation.