Linear Algebra
Exam 2 - Fall 2024

November 14, 2024

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid.

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

1. [5 points] Give an example of two 2×2 matrices A and B with entries restricted to $1'$s and $2'$s that shows that $det(A) + det(B)$ is not equal to $det(A + B)$. Justify your solutions via calculations.

There are many choices that work and a few that don't.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, then $\det A = 2 - 1 = 1$

Let $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $\det B = 1 - 1 = 0$

$A + B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$ and $\det(A+B) = 6 - 4 = 2$

However, $1 + 0 \neq 2$.

2. [10 points] Compute the determinant of the following matrix by any method of your choice. Name the method(s) that you employ.

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

I will do co-factor expansion across row 1.

$$\det A = 1 . \det \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{co-factor expansion down col 1}} - 2 \det \underbrace{\begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}}_{\text{co-factor expansion down col 1.}}$$

$$= 1 \cdot \left(1 \cdot \det \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right) - 2 \left(2 \det \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\right)$$

$$= 1 \cdot 1 \cdot ((1 - 0) \quad - 2 \cdot 2 \; (4 - 0)$$

$$= 1 \quad - \quad 16 \quad = -15$$

3. [5 points] Based upon your answer to the previous question, is there a non-trivial solution to $A\mathbf{x} = \mathbf{0}$? Why or why not? Is there a non-trivial solution to $A^T\mathbf{x} = \mathbf{0}$? Why or why not?

As the determinant of A is not zero, by the Invertible Matrix Theorem ~~$A\vec{x}$~~ A is invertible. Also, $A\vec{x} = \vec{0}$ has only the trivial solution. So, answer is no.

As the determinant of A is -15, by Theorem 5 of Chapter 3, det A^T is -15, too. Thus, as above, $A^T\vec{x} = \vec{0}$ has only the trivial solution. So, again, no.

4. [5 points] Show that the following set is *not* a subspace of $\mathbb{R}^3$ by showing that it fails to have at least two of the necessary properties.

$$H = \left\{ \begin{bmatrix} s \\ t \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

(1) H does not contain $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ since every vector in H has a 1 in the third entry.

(2) H is not closed under addition as

$$\begin{bmatrix} s \\ t \\ 1 \end{bmatrix} + \begin{bmatrix} s' \\ t' \\ 1 \end{bmatrix} = \begin{bmatrix} s+s' \\ t+t' \\ 2 \end{bmatrix}.$$

The sum of two vectors in H is not in H as the sum has a 2 in the third entry, not a 1.

(3) H is not closed under scalar multiplication as

$$c\begin{bmatrix} s \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} cs \\ ct \\ c1 \end{bmatrix}.$$

The result is a vector with c in the third entry and if $c \neq 1$, this vector

5. [5 points] Find two distinct bases for the set of all vectors of the form

$$\begin{bmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{bmatrix}.$$

This set of vectors is the set of all vectors of the form $a\begin{bmatrix}1\\2\\-1\\3\end{bmatrix} + b\begin{bmatrix}-2\\5\\-4\\1\end{bmatrix} + c\begin{bmatrix}5\\-8\\7\\1\end{bmatrix}$.

So, we seek two distinct bases for the column space of A where $A = \begin{bmatrix}1 & -2 & 5\\ 2 & 5 & -8\\ -1 & -4 & 7\\ 3 & 1 & 1\end{bmatrix}$. So, we ask, are these 3 vectors forming a linearly independent set? Let's Do G.E.!!

$$\begin{bmatrix}1 & -2 & 5\\ 2 & 5 & -8\\ -1 & -4 & 7\\ 3 & 1 & 1\end{bmatrix} \sim \underset{\substack{-2R1+R2\\ R1+R3\\ -3R1+R4}}{\begin{bmatrix}1 & -2 & 5\\ 0 & 9 & -18\\ 0 & -6 & 12\\ 0 & 7 & -14\end{bmatrix}} \sim \underset{\frac{1}{9}R2}{\begin{bmatrix}1 & -2 & 5\\ 0 & 1 & 2\\ 0 & -6 & 12\\ 0 & 7 & -14\end{bmatrix}} \sim \begin{bmatrix}1 & -2 & 5\\ 0 & 1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\Rightarrow \dim \operatorname{Col} A = 2$.

So columns 1 and 2 form a linearly independent set, thus a basis. But so do columns 1 and 3.

6. [10 points] Let A be a matrix of size 5×8. Suppose that upon row reducing A we find that there are precisely 3 pivot columns.

(a) What is the dimension of the null space?

The # of pivot columns equals the rank of A.
By the rank theorem, $\text{rank } A + \dim \text{Nul } A = n$

$\Rightarrow \quad 3 + \dim \text{Nul } A = 8$

$\Rightarrow \dim \text{Nul } A = 5$

(b) The null space is a subspace of which vector space?

It is a subspace of the domain, which in this case is $\mathbb{R}^8$.

(c) Let p denote the answer you gave in Part (a). Is the null space equal to $\mathbb{R}^p$? Why or why not?

So, $p = 5$.
No, it is not equal to $\mathbb{R}^5$ but rather it is isomorphic to $\mathbb{R}^5$. It is a subspace of $\mathbb{R}^8$.

(d) Consider a set of $p + 1$ vectors in Nul$A$. State one fact about this set.

Any $p+1$ vectors in a p-dimensional space are automatically linearly dependent.

(e) What is the rank of matrix A?

We said it already: $\text{rank } A = 3$.

7. [5 points] The following matrix P is not a regular stochastic matrix.

$$\begin{bmatrix} 0.66 & 0.5 \\ 0.34 & 0.66 \end{bmatrix}$$

Fix/correct/replace one of the entries so that it is. Set up but do not solve the equation that finds the steady-state vector for the Markov chain. What technique would allow you to solve this equation?

A regular stochastic matrix is a matrix in which the sum of the entries in any given column is 1. We can see that the 1st column's entries has this property but the second does not. So, there are two possible answers here either change to $\begin{bmatrix} .66 & .5 \\ .34 & .5 \end{bmatrix}$ or $\begin{bmatrix} .66 & .34 \\ .34 & .66 \end{bmatrix}$.

To find the steady-state vector we use Gaussian Elimination on the system $P\vec{x} = 1\vec{x}$. After this we may scale the vector $\vec{x}$ appropriately so that its entries sum to 1.

This is Exercise 14 in Section 4.7.

8. [10 points] In $\mathbb{P}^2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1-3t^2, 2+t-5t^2, 1+2t\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

By Theorem 15, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[[\vec{b}_1]_{\mathcal{C}} \; [\vec{b}_2]_{\mathcal{C}} \; [\vec{b}_3]_{\mathcal{C}} \right]$.

We first can "rename" these vectors as

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

and $$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Then if we do G.E. on

$\left[\vec{c}_1 \; \vec{c}_2 \; \vec{c}_3 \mid \vec{b}_1 \; \vec{b}_2 \; \vec{b}_3 \right]$ we will obtain $\left[I \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \right]$

But we're "already there" and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$

To write t^2 in terms of $\mathcal{B}$: first write t^2 as $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then solve $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{array}\right] \sim \cdots \cdots \sim \left[I \;\middle|\; \begin{matrix} 3 \\ -2 \\ 1 \end{matrix} \right].$

Thus $t^2 = 3\vec{b}_1 - 2\vec{b}_2 + 1\vec{b}_3$

$= 3(1-3t^2) - 2(2+t-5t^2) + (1+2t)$