Linear Algebra
Exam 2 - Fall 2024

November 14, 2024

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

1. [5 points] Give an example of two 2×2 matrices A and B with entries restricted to $1'$s and $2'$s that shows that $det(A) + det(B)$ is not equal to $det(A + B)$. Justify your solutions via calculations.

2. [10 points] Compute the determinant of the following matrix by any method of your choice. Name the method(s) that you employ.

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

3. [5 points] Based upon your answer to the previous question, is there a non-trivial solution to $A\mathbf{x} = \mathbf{0}$? Why or why not? Is there a non-trivial solution to $A^T\mathbf{x} = \mathbf{0}$? Why or why not?

4. [5 points] Show that the following set is *not* a subspace of $\mathbb{R}^3$ by showing that it fails to have at least two of the necessary properties.

$$H = \{ \begin{bmatrix} s \\ t \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \}$$

5. [5 points] Find two distinct bases for the set of all vectors of the form

$$\begin{bmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ 3a + b + c \end{bmatrix}.$$

6. [10 points] Let A be a matrix of size 5×8. Suppose that upon row reducing A we find that there are precisely 3 pivot columns.

(a) What is the dimension of the null space?

(b) The null space is a subspace of which vector space?

(c) Let p denote the answer you gave in Part (a). Is the null space equal to $\mathbb{R}^p$? Why or why not?

(d) Consider a set of $p + 1$ vectors in Nul$A$. State one fact about this set.

(e) What is the rank of matrix A?

7. [5 points] The following matrix P is not a regular stochastic matrix.

$$\begin{bmatrix} 0.66 & 0.5 \\ 0.34 & 0.66 \end{bmatrix}$$

Fix/correct/replace one of the entries so that it is. Set up but do not solve the equation that finds the steady-state vector for the Markov chain. What technique would allow you to solve this equation?

8. [10 points] In $\mathbb{P}^2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.