Linear Algebra
Exam 2 - Fall 2023

November 9, 2023

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

1. [10 points] Compute the determinant of the following matrix first by co-factor expansion (across a row or down a column of your choosing), then second by using Gaussian Elimination. (Next page is blank to give space for your neatly presented computations.)

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

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2. [5 points] Based upon your answer to the previous question, is the set of column vectors contained in A linearly independent? Why or why not?

3. [5 points] Based upon your answer from that same problem about the determinant of A, is there some integer k such that the $k-$th power of A has determinant 0? That is, if we take higher and higher powers of A (like $A^2, A^3, A^4 \ldots$), is it possible that for some power k we have $det(A^k) = 0$?

4. [10 points] Find a basis for the row space of A, the column space of A and the null space of A for the following matrix A.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

5. [5 points] State the dimensions of each of the subspaces that you just found. Next, the mapping $\mathbf{x} \mapsto A\mathbf{x}$ has a domain and a co-domain. In which is the column space located? In which is the null space located?

6. [5 points] The following set of vectors is not a basis for $\mathbb{R}^3$. Say why this is the case. Then find a basis for $\mathbb{R}^3$ that contains this set and say why the set you have constructed has the desired property.

$$\mathbf{b_1} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

7. [5 points] The following statements are false. Give a counter-example to each.

(a) If there exists a linearly dependent set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in V, then $dim V \leq p$.

(b) A linearly independent set in a subspace H is a basis for H.

8. [5 points] Let $\mathcal{D} = \{\mathbf{d_1}, \mathbf{d_2}, \mathbf{d_3}\}$ and $\mathcal{F} = \{\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}\}$ be bases for a vector space V, and suppose $\mathbf{f_1} = 2\mathbf{d_1} - \mathbf{d_2} + \mathbf{d_3}, \mathbf{f_2} = 3\mathbf{d_2} + \mathbf{d_3}$ and $\mathbf{f_3} = -3\mathbf{d_1} + 2\mathbf{d_3}$. Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$. Then find $[\mathbf{x}]_{\mathcal{D}}$ for $\mathbf{x} = \mathbf{f_1} - 2\mathbf{f_2} + 2\mathbf{f_3}$.