Linear Algebra
Exam 1
Fall 2024

October 17, 2024

Name: Solution Key

Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Additional Statement: I have not observed another violating the Honor Code.

Signature: C. F. Gauss

Average $\frac{49.5}{60}$ for 54 students. SD 7.7

Directions: Complete all problems. Justify all answers/solutions. Calculators, notes or texts are not permitted. Cell phones should not be used at any time (even to check the time) - please put them away! There is a two-hour time limit. This exam is proctored by permission of the Dean of Faculty. Note that some questions have a writing limit.

1. [10 points] Use Gaussian Elimination to solve the following system of linear equations. Use parametric vector form to describe the solution set. Give a geometric description of the solution set.

$$-x_1 - x_2 = 1$$

$$x_1 + x_2 - x_3 + x_4 = -1$$

$$x_1 + x_3 = 1$$

We use Gaussian Elimination on the following augmented matrix.

$$\left[\begin{array}{cccc|c} -1 & -1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 1 \end{array}\right] \sim \underset{\substack{R1+R2 \\ R1+R3 \\ (-1)R1}}{\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 2 \end{array}\right]} \sim \underset{R2 \leftrightarrow R3}{\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]}$$

$$\sim \underset{\substack{(-1)R2 \\ (-1)R3}}{\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]} \sim \underset{R3+R2}{\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]} \sim \underset{-R2+R1}{\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]}$$

$\Rightarrow$ $x_1 = 1 - x_4$, $x_2 = -2 + x_4$, $x_3 = x_4$, x_4 free

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This is a line in $\mathbb{R}^4$ through the point $(+1,-2,0,0)$ in the direction of $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

2. [5 points] Consider the coefficient matrix A of the previous problem. Without doing any further calculation, is the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ in the span of the columns of A? How do you know this? (1 sentence writing limit)

Yes! As there is a pivot in every row of A, the columns of A span $\mathbb{R}^3$, which includes $\vec{b}$

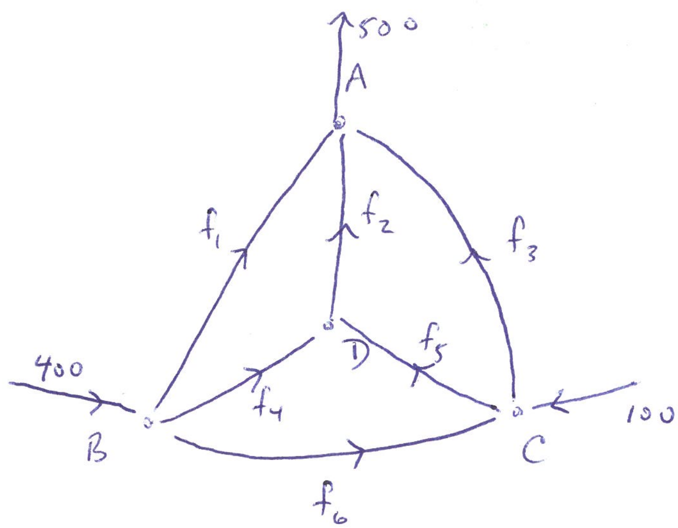
3. [5 points] Consider the coefficient matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$ of the two previous problems. Theorem 7 of Chapter 1 of David Lay's text gives a characterization of linearly dependent sets. That theorem states that the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ will be linearly dependent if and only if at least one of these is a linear combination of others (This fill-in-the-blank should *not* be filled in with our definition of linearly dependent set.). Which is the smallest j, $1 \leq j \leq 4$ for which a_j is a linear combination of the preceding vectors? How do you know this? (1 sentence writing limit)

Columns 1, 2, 3 are corresponding to basic variables, and column 4 to a free variable. Thus, $\vec{a}_4$ is a linear combination of the preceding ones, and thus 4 is the smallest index j for which this holds.

4. [10 points] **Network flow** Let us make the assumption of the preservation of flow in a network. Consider the following system of equations that is generated under this assumption by some network of 4 nodes.

$$f_1 + f_2 + f_3 = 500$$
$$f_1 + f_4 + f_6 = 400$$
$$f_3 + f_5 - f_6 = 100$$
$$f_2 - f_4 - f_5 = 0$$

Draw the network that generated this system. Be sure to label all nodes and arcs (i.e. label all vertices and directed edges to indicate direction of flow).



We can associate the first equation w/ node A, the second w/ node B, etc. Equation 3 is best rewritten as $f_3 + f_5 = 100 + f_6$ and Equation 4 as $f_2 = f_4 + f_5$. We then have one of two choices: the left-side of Equation 1 is "flow in" OR it is "flow out". I chose "flow in" for the diagram. The rest of the arc directions are forced.

5. [5 points] The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that shifts the plane to the right by one unit is not a linear transformation despite the fact that it takes equally-spaced lines to equally-spaced lines. (That is, the transformation is $T(x, y) = (x + 1, y)$.) Why isn't it a *linear* transformation? (2 sentence writing limit)

It is not a linear transformation since the origin does not map to the origin, i.e. $T(0,0) = (1,0) \neq (0,0)$.

OR

We show it fails to preserve vector addition:

$$\left.\begin{array}{l} T(x_1, y_1) + T(x_2, y_2) = (x_1 + x_2 + 2, y_1 + y_2) \\ T(x_1 + x_2, y_1 + y_2) = (x_1 + x_2 + 1, y_1 + y_2) \end{array}\right\} \text{not equal!}$$

6. [5 points]. Consider the coefficient matrix A of the first problem and the coefficient matrix F given in the problem on network flow. One of the two products AF and FA is defined - which one? For the one that is defined, give the size of the resulting matrix and use the row-column rule for multiplication to give the one entry in row 2, column 2.

Matrix A is 3×4 and matrix F is 4×6, so only AF is defined.

The entry in row 2, column 2 of matrix AF is found by multiplying row 2 of A with column 2 of F: this is

$$AF = \underbrace{\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & -1 & 1 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}}_{3\times 4} \underbrace{\begin{bmatrix} \vdots & 1 & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & 1 & \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{4\times 6} = \underbrace{\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 2 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & & & & \end{bmatrix}}_{3\times 6}$$

The resulting matrix has size 3×6.

7. [5 points] Professor Gauss (b. 1777 – d. 1855) did an elementary row operation on the matrix M below. What elementary row operation did he do? Instead of doing this elementary row operation, he could have multiplied M by what elementary matrix E to obtain the same matrix? In doing this, does Carl Friedrich compute EM or ME?

$$M = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 2 \\ 3 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

Gauss did $-3R1 + R3$, -3 times row 1 added to row 3.

He could have computed EM where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

8. [5 points] Suppose further that Professor Gauss finds out there is no set of elementary matrices that when multiplying M produces the identity matrix I_3. What does this imply about M? Give 2 facts.

This means that M is not invertible. So, by the IMT there are many other things we can say, for instance: the columns of M are linearly dependent

the columns of M do not span $\mathbb{R}^3$

M does not have 3 pivots, ...

9. [10 points] The LU-factorization of the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

is given by

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$$

Use the LU-factorization to **solve** the matrix equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 10 \\ 6 \\ 3 \end{bmatrix}$.

Check your solution.

The advantage of having been given the LU-factorization to solve the matrix equation is that (fill-in-the-blank) we have fewer flops to perform.

As $A = LU$ solving $A\vec{x} = \vec{b}$ is equivalent to solving $(LU)\vec{x} = \vec{b}$, or solving $L(U\vec{x}) = \vec{b}$. Let $\vec{y} = U\vec{x}$, then $L\vec{y} = \vec{b}$. We first solve $L\vec{y} = \vec{b}$ using G.E. as follows.

$$[L \mid \vec{b}] = \left[\begin{array}{ccc|c} 2 & 0 & 0 & 10 \\ 1 & -1 & 0 & 6 \\ -1 & 2 & 5 & 3 \end{array}\right] \underset{\frac{1}{2}R1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 1 & -1 & 0 & 6 \\ -1 & 2 & 5 & 3 \end{array}\right] \underset{\substack{-R1+R2 \\ R1+R3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 1 \\ 0 & 2 & 5 & 8 \end{array}\right] \underset{-1R2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 5 & 8 \end{array}\right]$$

$$\underset{-2R2+R3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 5 & 10 \end{array}\right] \underset{\frac{1}{5}R3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]. \quad \text{So, } \vec{y} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}.$$

We now solve $U\vec{x} = \vec{y}$ using G.E.:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim$$

$$\underset{\substack{+R3+R2 \\ -R3+R1}}{\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]} \underset{-2R2+R1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right], \quad \text{So } \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Check: $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 3 \end{bmatrix}$