Linear Algebra
Exam 1
Fall 2024

October 17, 2024

Name:
Honor Code Statement:

Additional Statement: I have not observed another violating the Honor Code.

Signature: ____________

Directions: Complete all problems. Justify all answers/solutions. Calculators, notes or texts are not permitted. Cell phones should not be used at any time (even to check the time) - please put them away! There is a two-hour time limit. This exam is proctored by permission of the Dean of Faculty. Note that some questions have a writing limit.

1. [10 points] Use Gaussian Elimination to solve the following system of linear equations. Use parametric vector form to describe the solution set. Give a geometric description of the solution set.

$$-x_1 - x_2 = 1$$
$$x_1 + x_2 - x_3 + x_4 = -1$$
$$x_1 + x_3 = 1$$

2. [5 points] Consider the coefficient matrix A of the previous problem. Without doing any further calculation, is the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ in the span of the columns of A? How do you know this? (1 sentence writing limit)

3. [5 points] Consider the coefficient matrix $A = [\mathbf{a_1} \ \ \mathbf{a_2} \ \ \mathbf{a_3} \ \ \mathbf{a_4}]$ of the two previous problems. Theorem 7 of Chapter 1 of David Lay's text gives a characterization of linearly dependent sets. That theorem states that the set $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}\}$ will be linearly dependent if and only if at least ____. (This fill-in-the-blank should *not* be filled in with our definition of linearly dependent set.). Which is the smallest j, $1 \leq j \leq 4$ for which a_j is a linear combination of the preceding vectors? How do you know this? (1 sentence writing limit)

4. [10 points] **Network flow** Let us make the assumption of the preservation of flow in a network. Consider the following system of equations that is generated under this assumption by some network of 4 nodes.

$$f_1 + f_2 + f_3 = 500$$
$$f_1 + f_4 + f_6 = 400$$
$$f_3 + f_5 - f_6 = 100$$
$$f_2 - f_4 - f_5 = 0$$

Draw the network that generated this system. Be sure to label all nodes and arcs (i.e. label all vertices and directed edges to indicate direction of flow).

5. [5 points] The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that shifts the plane to the right by one unit is not a linear transformation despite the fact that it takes equally-spaced lines to equally-spaced lines. (That is, the transformation is $T(x, y) = (x + 1, y)$.) Why isn't it a *linear* transformation? (2 sentence writing limit)

6. [5 points]. Consider the coefficient matrix A of the first problem and the coefficient matrix F given in the problem on network flow. One of the two products AF and FA is defined - which one? For the one that is defined, give the size of the resulting matrix and use the row-column rule for multiplication to give the one entry in row 2, column 2.

7. [5 points] Professor Gauss (b. 1777 – d. 1855) did an elementary row operation on the matrix M below. What elementary row operation did he do? Instead of doing this elementary row operation, he could have multiplied M by what elementary matrix E to obtain the same matrix? In doing this, does Carl Friedrich compute EM or ME?

$$M = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 2 \\ 3 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

8. [5 points] Suppose further that Professor Gauss finds out there is no set of elementary matrices that when multiplying M produces the identity matrix I_3. What does this imply about M? Give 2 facts.

9. [10 points] The LU-factorization of the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

is given by

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$$

Use the LU-factorization to **solve** the matrix equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 10 \\ 6 \\ 3 \end{bmatrix}$.

Check your solution.

The advantage of having been given the LU-factorization to solve the matrix equation is that (fill-in-the-blank) ____.