Graph Theory - MATH 247

Exam 2

Name:
Honor Code Pledge:

Signature:

Directions: Please complete all but 1 problem. There is a time limit of 3 hours.

1. Prove that every nontrivial tree has at least two maximal independent sets, with equality only for stars. (Note: maximal $\neq$ maximum.)
2. Prove that a d-regular simple graph G has a decomposition into copies of $K_{1,d}$ if and only if it is bipartite.
3. For $k \geq 2$, prove that the k-dimensional hypercube, Q_k, has at least $2^{2^{k-2}}$ perfect matchings. Next, determine for which k equality does NOT hold. (Earn an A for the course by determining the number of perfect matchings exactly - but don't do it now.)
4. Given the tree, T, in Figure 1, show EXPLICITLY how K_8 decomposes into 7 copies of this tree. (Colored pens are available at the front of the room to help, and a convenient sketch of K_8 is attached - see Figure 5. And, yes, I want them back.)

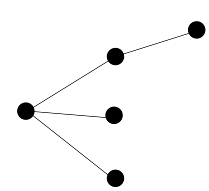

Figure 1: A four edge tree T

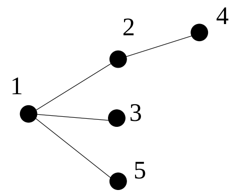

Figure 2: A four edge vertex-labelled tree

5.
   - Given the vertex-labelled tree in Figure 2 determine its Prüfer code.
   - Given the following Prüfer code $(1, 1, 1, 3, 3, 3)$, determine the tree which corresponds to it.

6.
   - Find a minimum weight spanning tree of the graph in Figure 3. (Use the first copy attached.)
   - Apply Dijkstra's algorithm to obtain the distance from vertex u to each of the other vertices in the graph in Figure 4. (Use the second copy attached.)

7. Without using any results on matchings in bipartite graphs, prove directly that every regular bipartite graph with positive degree satisfies Tutte's Condition (and therefore, by Tutte's Theorem, has a perfect matching).

8. A deck of mn cards with m values and n suits consists of one card of each value in each suit. THe cards are dealt into an $n-$by-m array. Prove that there is a set of m cards, one in each column, having distinct values. (Hint (which you'll probably try to ignore): form a X, Y-bigraph in which X represents the columns and Y represents the values, with r edges from $x \in X$ to $y \in Y$ if value y appears r times in columns x.)

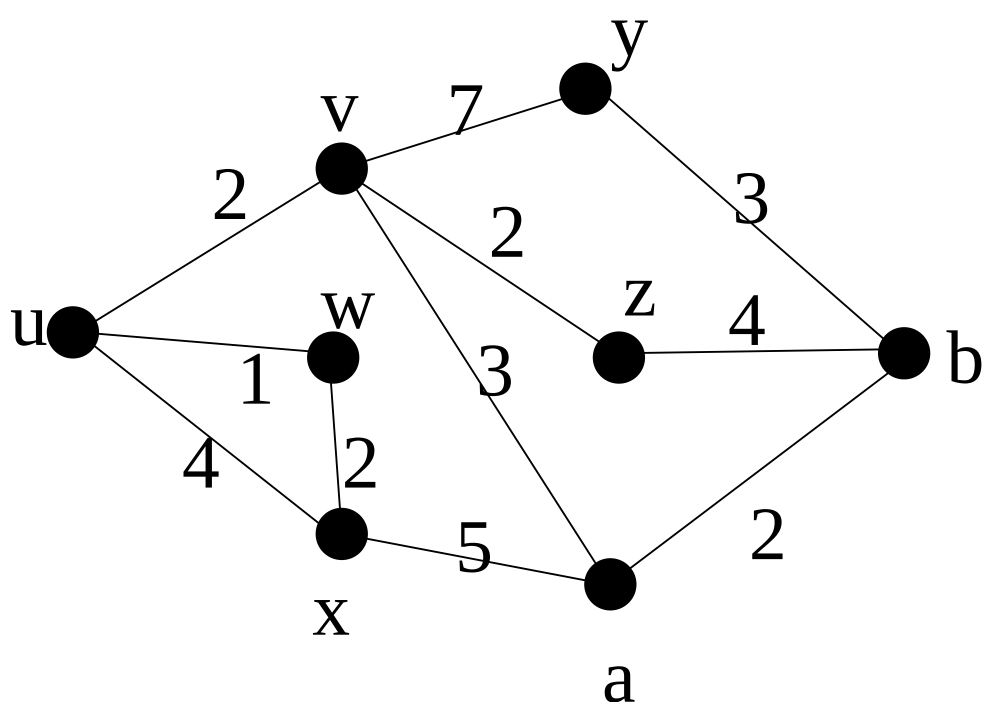

Figure 3: A graph, G

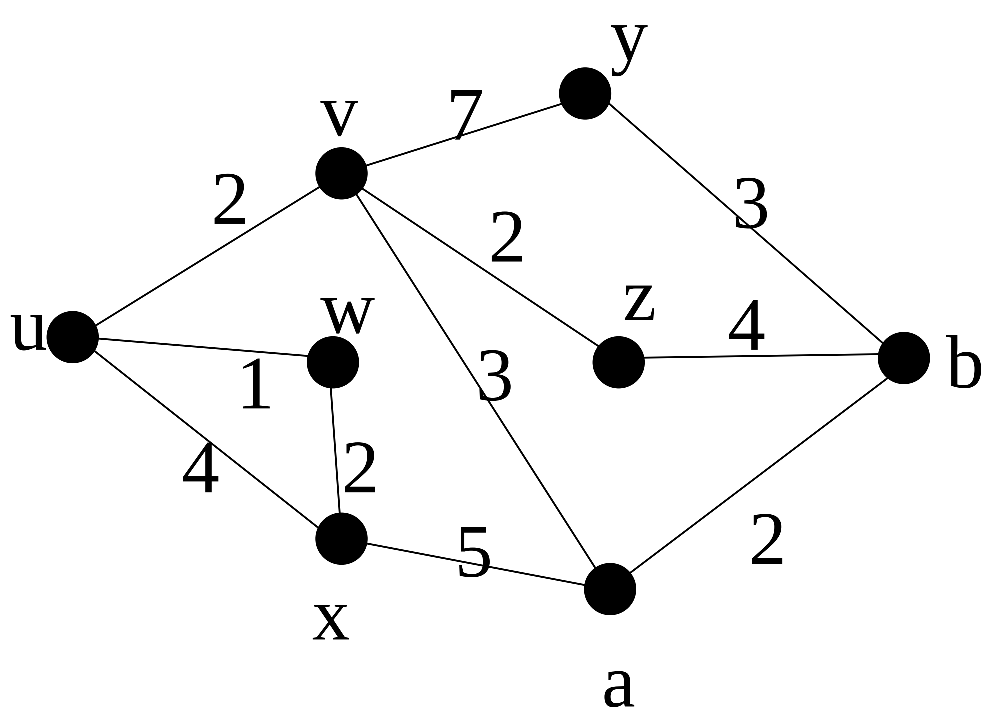

Figure 4: A graph, G

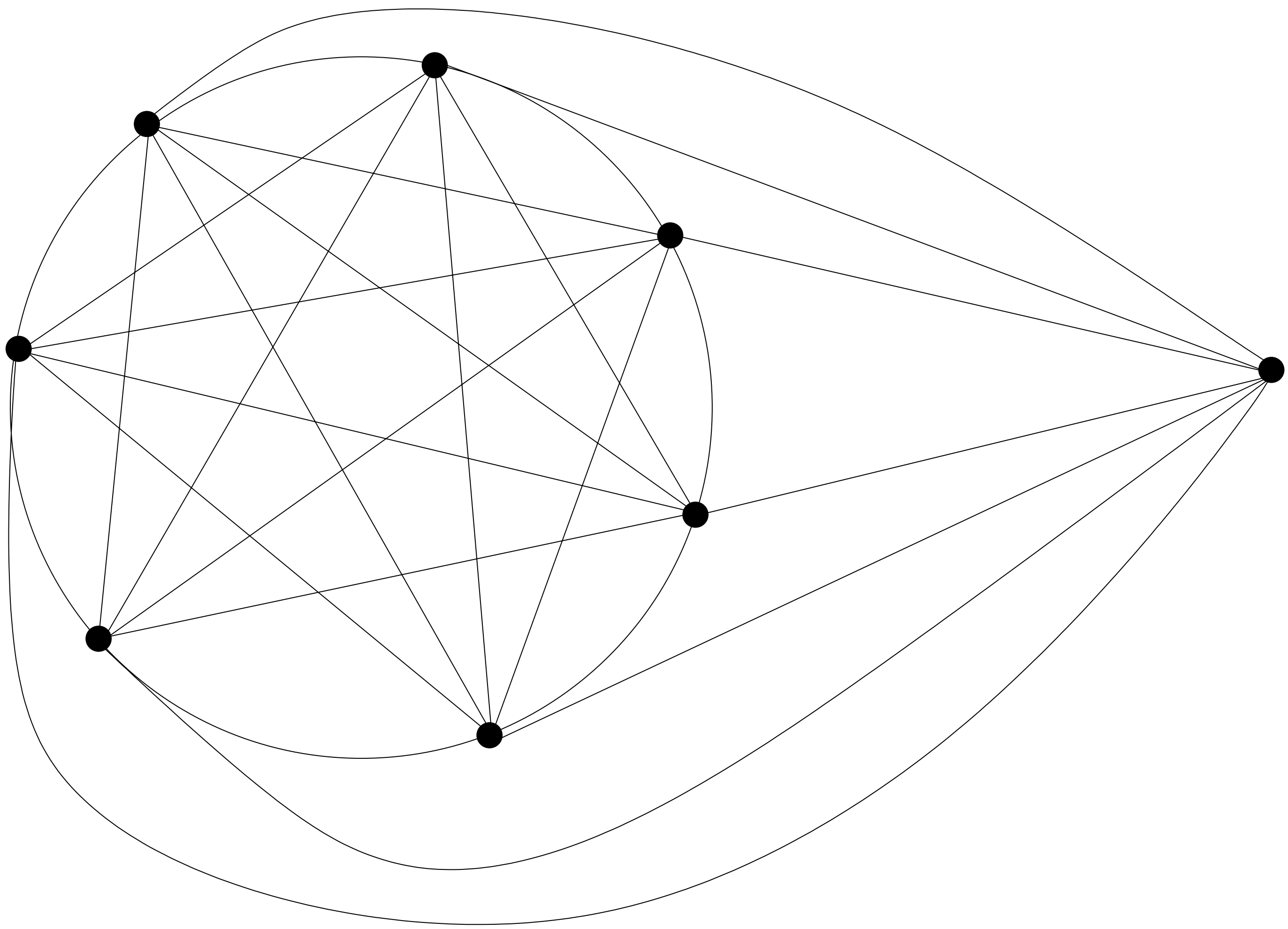

Figure 5: The complete graph on 8 vertices, K_8