Name ____
ID number_____________________________________
Sections **C and D**

Calculus II (Math 122) Final Exam, 19 May 2012

This is a closed book exam. No notes or calculators are allowed. A table of trigonometric identities is attached. To receive credit you must show your work. Please leave answers as square roots, $\ln(), \exp()$, fractions, or in terms of constants like e, π, etc. Please turn off all cell-phones and other electronic devices. When you are finished please write and sign the honor code, (I have neither given nor received unauthorized aid on this exam) in the space provided below. Please remember that you are also obligated to report violations of the honor code. Good luck!

1	
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Total	

Honor Code:

Signature:

1. [3 each] State clearly and precisely the following:

   (a) Definition of an infinite sequence (using the language of functions)

   (b) Definition of a geometric series

   (c) Definition of a continuous function, $f(x)$, at $x=c$ where c is in the domain of f

2. [2 points each] **Fill in the blank** Please complete the proof by justifying why each step is true.

We will solve the integral equation

$$y(x) = 2 + \int_1^x \frac{dt}{ty(t)}, \quad x > 0.$$

We begin by ____________ to obtain $y'(x) = \frac{1}{xy(x)}$.

Note that we know the derivative of the right side by applying ____.

We now have $\frac{dy}{dx} = \frac{1}{xy}$, which is a ________ differential equation.

So we may write, $\int y dy = \int \frac{1}{x} dx$.

From this we obtain $\frac{1}{2}y^2 = \ln(x) + C, \quad x > 0$. The right-side of this equation is the result of knowing __________________________.

Letting $x = 1$ in the original equation, we find that $y(1) =$ ________. Thus C equals ___ and we are able to solve for y. $\square$

3. [9 points each] Determine whether the following series converge or diverge. You must state or clearly demonstrate what test you are using to determine convergence and justify its use. Heuristic or intuitive reasoning will not get full credit, though it may help you get started.

   (a) $\sum_{n=1}^{\infty} n^2 3^{-n^2}$

(b) $\sum_{n=1}^{\infty} \frac{3-n^2}{(n+3)^3}$

(c) $\sum_{n=1}^{\infty} (\frac{n!}{n^n})^2$

4. [10 points] Determine the interval of convergence for the power series.

$$\sum_{k=0}^{\infty}(-1)^k\frac{(x-5)^k}{3^k(k+1)}$$

5. **Taylor's method**

(a) [10 points] Determine the first four terms of the Taylor series of $f(x) = \sqrt{x}$ at $x = 1$.

(b) [8 points] Use the third-degree Taylor polynomial to *show how* to approximate $\sqrt{1.5}$ as a decimal. (You needn't do the arithmetic.)

(c) [7 points] Use Taylor's Inequality to bound the error for the estimate given in the previous step.

6. [8 points each] Evaluate each of the following integrals. State the domain of your answer.

(a) $\int \frac{x^2+6x+9}{x^2+6x} dx$

(b) $\int [\sec(x) - \tan(x)]^2 dx$

(c) $\int \frac{1}{\sqrt{9-4x^2}} dx$

(d) $\int \ln(2t)\frac{1}{t} dt$

7. [15 points]Carbon dating is a method used by scientists to accurately determine the age of organic matter that may be up to around $50,000$ years. When a plant or animal is living, the ratio of radioactive Carbon-14 to ordinary carbon stays constant. After the organism dies, no new carbon is ingested, and the proportion of Carbon-14 in the organisms remains decreases as the Carbon-14 decays. The rate of decay of Carbon-14 is proportional to the amount still present. The half-life of Carbon-14 is $5,700$ years.

   (a) Define some useful notation and give a differential equation that models the decay of Carbon-14. What initial condition would you hope to have to help make your model more specific?

   (b) Solve the differential equation to find the amount of radioactive Carbon-14 nuclei as a function to time. You may express your answer in terms of the initial amount of radioactive material present when the organism was still alive. Indicate how the stated half-life relates to your model and helps to make it more specific and thus more useful.

   (c) [Application] An ancient scroll was found by archeologists in the Middle East in a sealed earthen vase, and the well-preserved manuscript contained 70% of the Carbon-14 found in living matter. About how old is the manuscript?

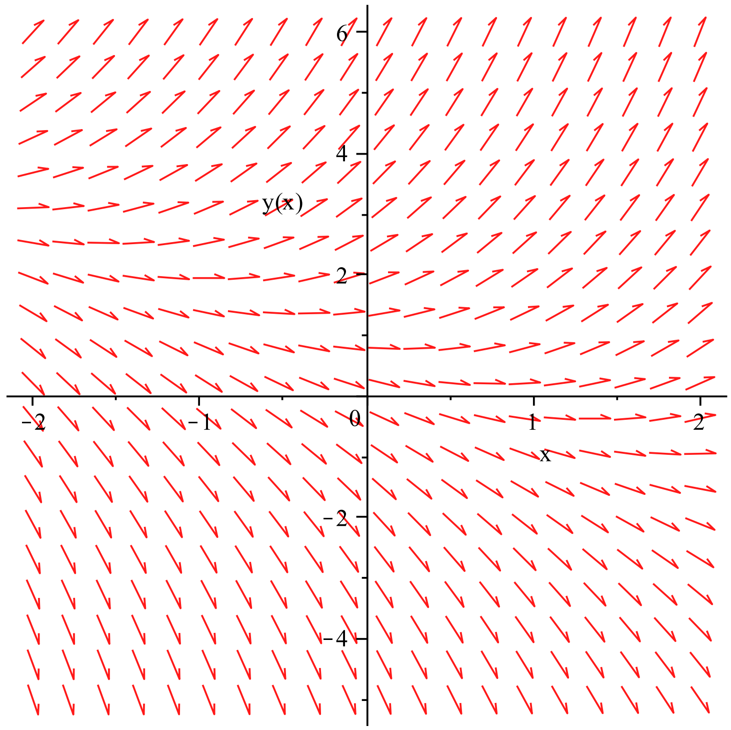

Figure 1: The direction field

8. [10 points] A direction field for the differential equation $y' = x + y - 1$ is shown.

   (a) Sketch the graphs of the solutions that satisfy the given initial condition $(-1, 2)$.

   (b) Use Euler's method with step size of $h = 0.2$ to estimate $y(-0.6)$ (using the same initial conditions as above).

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$