Name ____
ID number______________________________________
Section **A**

Calculus II (MATH 0122) Final Exam, 15 December 2022

This is a closed book exam. Notes, calculators, cell-phones are not allowed – the only allowable items are pens, pencils and erasers. A table of trigonometric identities is attached. To receive credit you must show your work. Please leave answers as square roots, $\ln(), \exp()$, fractions, or in terms of constants like e, π, etc. Please turn off all cell-phones and other electronic devices. When you are finished please write and sign the Honor Code (I have neither given nor received unauthorized aid on this exam. I have not witnessed another giving or receiving unauthorized aid.) in the space provided below. Good luck!

Please complete all of the questions.

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Total	

Honor Code:

Signature:

1. [5 points] Write $4.123123\ldots$ as the ratio of two integers.

2. [10 points] In this problem you will show that $\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3}$ converges.

First, let's show that $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ converges by using the Integral Test. Show this here (and to do so, you will need to use integration by parts):

Now let's note that $\frac{k\ln(k)}{(k+1)^3} < \frac{k\ln(k)}{k^3} = \frac{\ln(k)}{k^2}$. Explain why this inequality is true.

What test allows you to make the desired conclusion?

3. [10 points] Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0.$$

4. [10 points]

(a) Find the Taylor series for $f(x) = \ln(x)$ centered at the value of $a = 2$. [Assume that $f(x)$ has a power series expansion. Do not show that $R_n(x) \to 0$.]

(b) Find the associated radius of convergence.

5. [15 points] Populations of birds and insects are modeled by the equations

$$\frac{dx}{dt} = \frac{4}{10}x - \frac{2}{1,000}xy$$
$$\frac{dy}{dt} = -\frac{2}{10}y + \frac{8}{1,000,000}xy$$

(a) Which of the variables, x or y, represents the bird population and which represents the insect population?

(b) Find the equilibrium solutions and explain their significance. Mark these solutions on the plot given lower down.

(c) Find an expression for $\frac{dy}{dx}$.

(d) The direction field for the differential equation in Part (c) is shown. Use it to sketch a phase portrait.

6. [10 points]

(a) Use Euler's method with step size 0.2 to estimate $y(0.4)$, where $y(x)$ is the solution of the initial-value problem $y' = 2xy^2, \quad y(0) = 1$. (That is, find (x_2, y_2).)

(b) Find the exact solution of the differential equation and compare the value at 0.4 with the approximation in Part (a).

7. [15 points] Consider the following curve defined parametrically by

$$x = \sin(t), \quad y = \cos^2(t), \quad 0 \leq t \leq 2\pi.$$

(a) On the graph paper provided, sketch $x(t)$ on a t, x-plane and $y(t)$ on a t, y-plane.

(b) On the graph paper provided, sketch the curve on the x, y-plane.

(c) Find $\frac{dy}{dx}$.

(d) Find the equation of the line tangent to the curve at $t = \pi$ and draw the tangent line on the curve you sketched in Part (b).

Graph paper here.

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$