Name Solution Key
ID number
Section **A**

Calculus II (MATH 0122) Final Exam, 15 December 2022

This is a closed book exam. Notes, calculators, cell-phones are not allowed – the only allowable items are pens, pencils and erasers. A table of trigonometric identities is attached. To receive credit you must show your work. Please leave answers as square roots, ln(), exp(), fractions, or in terms of constants like e, π, etc. Please turn off all cell-phones and other electronic devices. When you are finished please write and sign the Honor Code (I have neither given nor received unauthorized aid on this exam. I have not witnessed another giving or receiving unauthorized aid.) in the space provided below. Good luck!

Please complete all of the questions.

1	5
2	10
3	10
4	10
5	15
6	10
7	15
Total	75

Average 56 of 75
Standard deviation 13.

Honor Code:

I have neither given nor received unauthorized aid. I have not witnessed another giving or receiving unauthorized aid.

Signature:

Erdős, Pal

1. [5 points] Write 4.123123... as the ratio of two integers.

4.123123 equals 4 plus 0.123123....

The latter can be considered as a geometric series with first term $\frac{123}{1000}$ and common ratio $r = \frac{1}{1000}$.

That is, $0.123123\ldots = 123 \cdot \frac{1}{1000} + 123 \cdot \left(\frac{1}{1000}\right)^2 + 123\left(\frac{1}{1000}\right)^3 + \ldots$

This has sum $\frac{a}{1-r} = \frac{123/1000}{1 - 1/1000} = \frac{123}{999}$

So $4 + \frac{123}{999} = \frac{3996}{999} + \frac{123}{999} = \frac{4119}{999}$

2. [10 points] In this problem you will show that $\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3}$ converges.

First, let's show that $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ converges by using the Integral Test. Show this here (and to do so, you will need to use integration by parts):

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x}\Big|_1^\infty - \int_1^\infty -\frac{1}{x^2}\,dx$$

Let $u = \ln x \quad dv = x^{-2}dx$

$du = \frac{1}{x} \quad v = -x^{-1}$

$$= \frac{-\ln x}{x} - \frac{1}{x}\Big|_1^\infty$$

$$= \lim_{b\to\infty} -\frac{\ln x}{x} - \frac{1}{x}\Big|_1^b = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b}\right) - \left(-\frac{\ln 1}{1} - \frac{1}{1}\right)$$

$$= 1$$

Thus converges.

Now let's note that $\frac{k\ln(k)}{(k+1)^3} < \frac{k\ln(k)}{k^3} = \frac{\ln(k)}{k^2}$. Explain why this inequality is true.

Since $(k+1)^3$ is bigger than k^3.

What test allows you to make the desired conclusion?

As $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ is bigger term-by-term than the given series and it converges, then by the Comparison Test the given converges.

3. [10 points] Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0.$$

Notice that the given power series is differentiable term-by-term (using the power rule)

$$\frac{d\left(\frac{(-1)^n x^{2n}}{(2n)!}\right)}{dx} = \frac{(-1)^n (2n)\, x^{2n-1}}{(2n)!} = \frac{(-1)^n x^{2n-1}}{(2n-1)!}$$

Differentiating again

$$\frac{d\left(\frac{(-1)^n x^{2n-1}}{(2n-1)!}\right)}{dx} = \frac{(-1)^n (2n-1) x^{2n-2}}{(2n-1)!} = \frac{(-1)^n x^{2n-2}}{(2n-2)!}$$

So $f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n-2)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}$

Thus, $f''(x) + f(x) = 0$.

That is, $\frac{d\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}{dx} = 0 - \frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots$

and $\frac{d\left(0 - \frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots\right)}{dx} = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots$

And so we see that f and f'' sum to zero.

OR Notice
$f(x) = \cos x$
so $f'(x) = -\sin x$
and $f''(x) = -\cos x$

This was a problem from one of the p-sets.

Section 11.10 #21

4. [10 points]

(a) Find the Taylor series for $f(x) = \ln(x)$ centered at the value of $a = 2$. [Assume that $f(x)$ has a power series expansion. Do not show that $R_n(x) \to 0$.]

The Taylor series has the form $\frac{f(a)}{0!} + \frac{f'(a)(x-a)^1}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$

So we find

$f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$

$f''(x) = -\frac{1}{x^2} \qquad f''(2) = -\frac{1}{2^2}$

$f'''(x) = \frac{+2}{x^3}$

$\vdots \qquad \vdots$

$f^{k}(x) = (-1)^{k+1}\frac{(k-1)!}{x^k} \qquad f^{k}(2) = (-1)^{k+1}\frac{(k-1)!}{2^k}$

Thus, the series is

$\frac{\ln 2}{0!} + \frac{1}{2}(x-2)^1 + -\frac{1}{2^2}\cdot\frac{1}{2!}(x-2)^2 + \frac{2}{2^3}\frac{1}{3!}(x-2)^3 + \cdots \frac{(-1)^{k+1}}{2^k}(k-1)!\frac{1}{k!}(x-2)^k$

(b) Find the associated radius of convergence.

We can use the Ratio Test and consider the limit of the ratio of consecutive terms

$$\lim_{k\to\infty}\left|\frac{\frac{(-1)^{k+2}k!}{2^{k+1}}\frac{1}{(k+1)!}(x-2)^{k+1}}{\frac{(-1)^{k+1}(k-1)!}{2^k}\frac{1}{k!}(x-2)^k}\right| = \lim_{k\to\infty}\left|\frac{k}{2}\cdot\frac{1}{k+1}(x-2)\right|$$

$$= \frac{|x-2|}{2}$$

Want this less than 1, so $\left|\frac{x-2}{2}\right| < 1$. That is,

$-2 < x-2 < 2 \iff 0 < x < 4$. Radius is 2.

5. [15 points] Populations of birds and insects are modeled by the equations

$$\frac{dx}{dt} = \frac{4}{10}x - \frac{2}{1,000}xy$$
$$\frac{dy}{dt} = -\frac{2}{10}y + \frac{8}{1,000,000}xy$$

(a) Which of the variables, x or y, represents the bird population and which represents the insect population?

Birds prey on insects. As the xy term is negative in the first equation and positive in the second, x represents insects y birds.

(b) Find the equilibrium solutions and explain their significance. Mark these solutions on the plot given lower down.

$$\frac{dx}{dt} = x\left(\frac{4}{10} - \frac{2}{1000}y\right) = 0$$

$$\frac{dy}{dt} = y\left(-\frac{2}{10} + \frac{8}{1,000,000}x\right) = 0$$

One is $(x, y) = (0, 0)$.

Otherwise

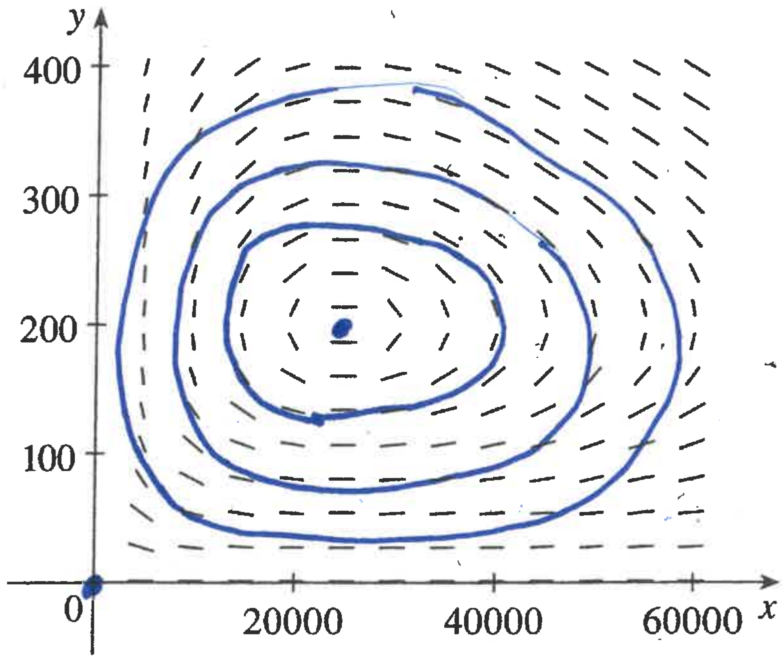
$$\frac{4}{10} - \frac{2}{1000}y = 0 \Rightarrow y = 200$$

$$-\frac{2}{10} + \frac{8}{1,000,000}x = 0 \Rightarrow x = 25000$$

(c) Find an expression for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2/10\,y + 8/1000000\,xy}{4/10\,x - \frac{2}{1000}xy}$$

(d) The direction field for the differential equation in Part (c) is shown. Use it to sketch a phase portrait.



equilibrium points marked as points

6. [10 points]

(a) Use Euler's method with step size 0.2 to estimate $y(0.4)$, where $y(x)$ is the solution of the initial-value problem $y' = 2xy^2$, $y(0) = 1$. (That is, find (x_2, y_2).)

(b) Find the exact solution of the differential equation and compare the value at 0.4 with the approximation in Part (a).

(a) $x_0 = 0$, $y_0 = 1$ is given.

$$y_1 = y_0 + h\,F(x_0, y_0) = 1 + 0.2\,(2 \cdot 0 \cdot 1^2) = 1$$

so $(x_1, y_1) = (0.2, 1)$

$$y_2 = y_1 + h\,F(x_1, y_1) = 1 + 0.2\,(2 \cdot 0.2 \cdot 1^2) = 1.08$$

so $(x_2, y_2) = (0.4, 1.08)$

(b) $\frac{dy}{dx} = 2xy^2$ is a separable differential equation.

$$\Rightarrow \frac{dy}{y^2} = 2x\,dx \Rightarrow \int \frac{dy}{y^2} = \int 2x\,dx$$

$$-\frac{1}{y} = x^2 + C$$

We can solve for C using the initial condition, $x_0 = 0$, $y_0 = 1$

So

$$\frac{-1}{1} = 0^2 + C \Rightarrow C = -1$$

Thus, $\frac{-1}{y} = x^2 - 1 \Rightarrow y = \frac{-1}{x^2 - 1}$

So at $x = 0.4$ we have $y = \frac{-1}{0.16 - 1} = \frac{100}{84} = \frac{25}{21} \cong 1.19$

The estimate in part (a) is under by ≈ 0.11.

7. [15 points] Consider the following curve defined parametrically by

$$x = \sin(t), \quad y = \cos^2(t), \quad 0 \le t \le 2\pi.$$

(a) On the graph paper provided, sketch $x(t)$ on a t, x-plane and $y(t)$ on a t, y-plane.

(b) On the graph paper provided, sketch the curve on the x, y-plane.

(c) Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos(t)\,(-\sin t)}{\cos(t)}$$

$$= -2\sin t$$

(d) Find the equation of the line tangent to the curve at $t = \pi$ and draw the tangent line on the curve you sketched in Part (b).

At $t = \pi$, the point on the curve is

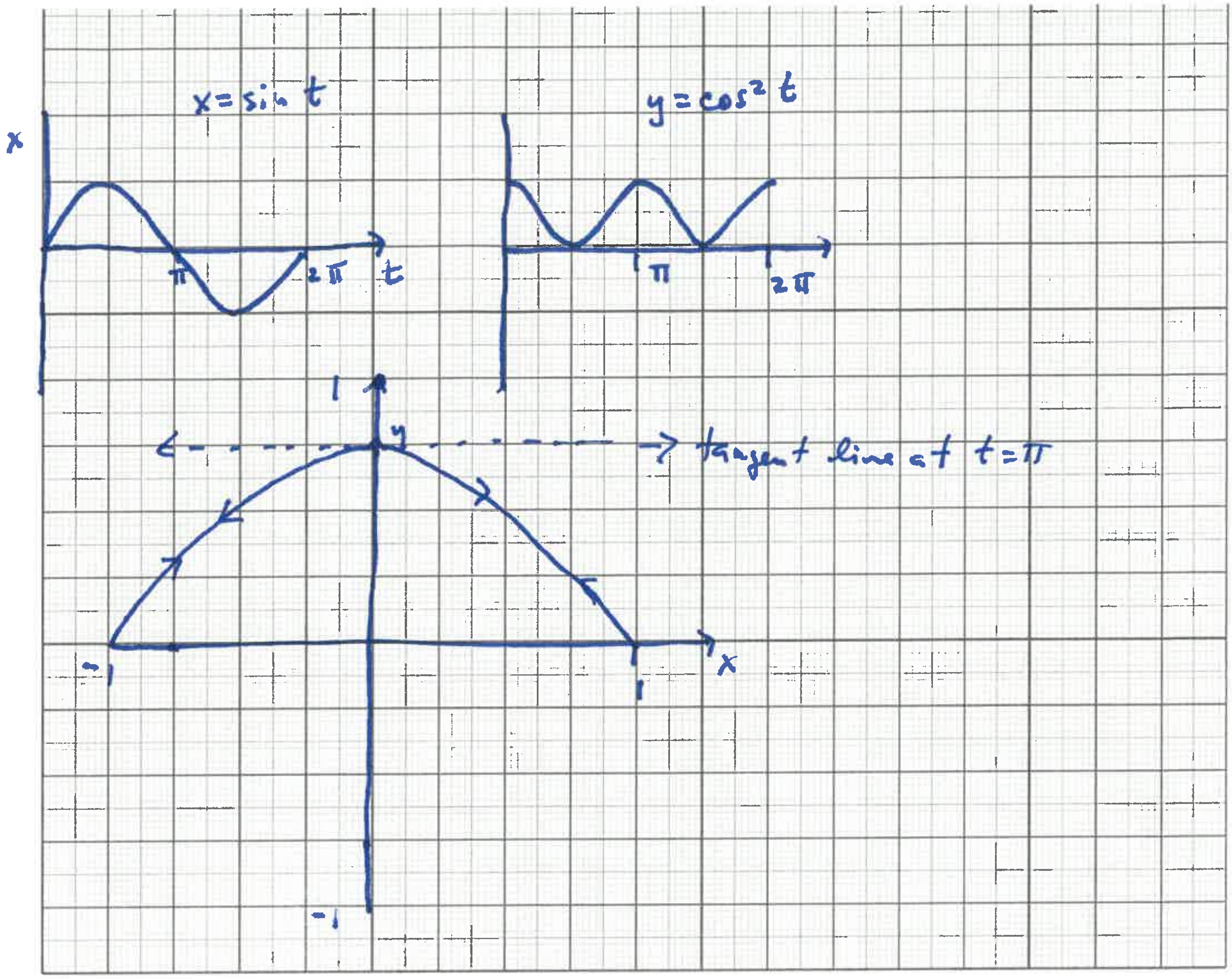
$x = \sin(\pi) = 0$ and $y = \cos^2(\pi) = 1$

At $t = \pi$, the slope of the tangent line at this point is $\left.\frac{dy}{dx}\right|_{t=\pi} = -2\sin(\pi) = 0$

Thus, the tangent line has equation

$$y - 1 = 0(x - 0)$$

$\Rightarrow$ $y = 1$, a horizontal line thru $(0,1)$



t	x	y
0	0	1
$\pi/2$	1	0
π	0	1
$3\pi/2$	-1	0
2π	0	1

The curve starts at (0,1) moves to (1,0) and then back to (0,1) tracing over itself. Then to (-1,0) and back over itself to (0,1).

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$