Calculus II - Exam 3 - Fall 2022

November 17, 2022

Name: Solution Key

Honor Code Statement: I have neither given nor received unauthorized aid.

Total points 60

Avg 44 points

Directions: Upon completion of the examination and prior to its submission, please write and sign the Honor Code. **Justify** all answers/solutions. Calculators are not permitted, and all electronic devices should be off. Good luck!

1. [5 points] Give an example of a bounded *sequence* that is monotonically decreasing and bounded from below by 3. Give another example of a *sequence* that diverges.

A frequent error is to give a series and not a sequence.

Example 1

$$\{a_n\} = \left\{3 + \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{4, 3\tfrac{1}{2}, 3\tfrac{1}{3}, 3\tfrac{1}{4}, \dots\right\}$$

We have a lower bound of 3 since $3 + \frac{1}{n} > 3$ for all $n \geq 1$.

It is decreasing since $3 + \frac{1}{n+1} < 3 + \frac{1}{n}$ for all $n \geq 1$.

Example 2

$$\{b_n\} = \{n\}_{n=1}^{\infty} = \{1, 2, 3, 4, \dots\}$$

For every positive number M, there is an integer $N = \lceil M \rceil + 1$ such that for $n > N$ we have $b_n > M$.

That is, say $M = 1{,}000{,}000$, then $b_{1{,}000{,}001} > 1{,}000{,}000$

2. [8 points each] **Complete each of the following using the indicated test.** For each of the following series, determine whether or not the series converges. If the series contains any negative terms, please test for absolute convergence.

(a) Use the Comparison Test

$$\sum_{n=1}^{\infty} \frac{5}{5n-1}$$

The degree of the numerator of a term is 0, and that of the denominator is 1. So, this reminds me of the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent p-series.

If we can show that the given series is larger term-by-term than $\sum_{n=1}^{\infty} \frac{1}{n}$, then the given series also diverges.

Note that $\frac{5}{5n-1} > \frac{1}{n}$ since $5n > 5n-1$ for $n \geq 1$.

Thus, the given series diverges.

(b) Use the Test for Divergence

$$\sum_{n=1}^{\infty} \frac{n+1}{n}$$

The test for divergence says that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Here: $\lim_{n\to\infty} \frac{n+1}{n} = \frac{\infty}{\infty}$, an indeterminate form to which we can apply L'Hôpital's Rule.

$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty} \frac{1}{1} = 1$. Thus, the given series diverges.

(c) Use the Ratio Test

$$\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$$

By the Ratio Test,
We consider $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Here $\lim_{n\to\infty} \dfrac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}} = \lim_{n\to\infty} \frac{1}{3} \frac{2^{n+1}+5}{2^n+5} = \frac{\infty}{\infty}$, an indeterminate form.

Apply L'Hôpital's Rule, this equals $\frac{1}{3} \lim_{n\to\infty} \frac{(\ln 2) 2^{n+1}}{(\ln 2) 2^n} = \frac{1}{3} \lim 2$

$= \frac{2}{3}$

As $\frac{2}{3} < 1$, the Ratio Test implies convergence.

(d) Use the Root Test

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

We consider $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}}$

$$= \lim_{n\to\infty} \frac{2}{n^{2/n}} = \frac{\lim_{n\to\infty} 2}{\lim_{n\to\infty} n^{2/n}}$$

We now compute $\lim_{n\to\infty} n^{2/n} = y$ separately.

$y = \lim_{n\to\infty} n^{2/n}$ is of the indeterminate form ∞^0

So, let

$$\ln y = \ln \lim_{n\to\infty} n^{2/n}$$

$$= \lim_{n\to\infty} \ln n^{2/n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \ln(n) = \frac{\infty}{\infty}$$

$$\overset{\text{L.H.}}{=} \lim_{n\to\infty} \frac{2 \cdot 1/n}{1} = 0 \implies \ln y = 0 \text{ and}$$

So $y = 1$

Thus $\lim \sqrt[n]{a^n} = \frac{2}{1}$, which is bigger than 1. Thus, by the Root Test, this series diverges.

(e) Use the Integral Test

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

The given series converges if and only if

$\int_1^{\infty} x^2 e^{-x^3}\,dx$ converges. This improper integral

can be solved using a u-substitution with $u=-x^3$, $du=-3x^2dx$
(Notice that the integrand is positive, continuous + decreasing.)
Thus,

$$\lim_{b\to\infty} \int_1^b x^2 e^{-x^3}\,dx = \lim_{b\to\infty} -\frac{1}{3}\int_1^b -3x^2 e^{-x^3}\,dx$$

$$= \lim_{b\to\infty} -\frac{1}{3} e^{-x^3}\Big|_1^b$$

$$= \lim_{b\to\infty} -\frac{1}{3} e^{-b^3} - \left(-\frac{1}{3} e^{-1}\right)$$

$$= \lim_{b\to\infty} -\frac{1}{3}\cdot\frac{1}{e^{b^3}} + \frac{1}{3}\cdot\frac{1}{e}$$

$$= 0 + \frac{1}{3e}$$

The integral converges. Thus the series converges.

3. [5 points] Express the repeating decimal 5.232323... as the ratio of two integers.

Writing this repeating decimal as a series, we get

$$5 + \frac{23}{100} + \frac{23}{10{,}000} + \frac{23}{1{,}000{,}000} + \cdots$$

$$= 5 + \sum_{n=1}^{\infty} \left(\frac{23}{100}\right) \cdot \left(\frac{1}{100}\right)^{n-1}$$

We recall that the sum of a geometric series w/ first term $a = \frac{23}{100}$ and common ratio $r = \frac{1}{100}$ has

Sum $\frac{a}{1-r} = \frac{23/100}{1 - 1/100} = \frac{23/100}{99/100} = \frac{23}{99}$

So we have $5.23\,23\,23 \ldots$ equal to $5\frac{23}{99} = \frac{518}{99}$.

4. [10 points] Find the Taylor Series for $f(x) = \cos x$ at $a = \frac{\pi}{2}$. Then use the third-degree Taylor polynomial to give an estimate for $\cos(2)$.

We use Taylor's method and begin by finding the derivatives of $\cos x$ and then evaluating these at $a = \frac{\pi}{2}$

$f(x) = \cos x \qquad f(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$f'(x) = -\sin x \qquad f'(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

$f''(x) = -\cos x \qquad f''(\frac{\pi}{2}) = -\cos\frac{\pi}{2} = 0$

$f'''(x) = \sin x \qquad f'''(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

and then these "loop" with a period of 4.

So the Taylor Series is $0 + \frac{(-1)}{1!}(x-\frac{\pi}{2})^1 + 0 + \frac{(+1)}{3!}(x-\frac{\pi}{2})^3 + 0 + \frac{(-1)}{5!}(x-\frac{\pi}{2})^5 + \ldots$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n-1)!}\left(x-\frac{\pi}{2}\right)^{2n-1}$$

Thus the 3rd-degree Taylor polynomial is

$$T_3(x) = \frac{-1}{1!}\left(x-\frac{\pi}{2}\right) + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3$$

Thus, an estimate for $\cos(2)$ is

$$T_3(2) = \frac{-1}{1!}\left(2-\frac{\pi}{2}\right) + \frac{1}{3!}\left(2-\frac{\pi}{2}\right)^3.$$