# Calculus II - Exam 2 - Techniques of Integration

October 20, 2022

**Name:**
**Honor Code Statement:**

**Additional Honor Statement:** I have not observed another violating the Honor Code.__________________________

**Directions: Justify** all answers/solutions. For each of the first five problems, find the integral and **identify** the technique of integration used. Calculators are not permitted. You may use the table of trigonometric identities given on the last page. Each problem is worth 10 points. If you need extra space, use the blank white paper provided.

1. Evaluate the integral.

$$\int x \sec^2(x)dx$$

We use integration by parts: $\int u\,dv = uv - \int v\,du$.

Let $u = x$ and $dv = \sec^2 x\,dx$. Then $du = 1\,dx$ and $v = \tan x$

Thus, $\int x \sec^2 x\,dx = x\tan x - \int \tan x\,dx$

$= x\tan x - \int \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x$ then $du = -\sin x\,dx$

Recall $\int \frac{du}{u} = \ln|u| + c$.

Thus, we obtain

$x\tan x + \ln|\cos x| + c$

2. Evaluate the integral.

$$\int \tan^2\theta \sec^4\theta d\theta$$

This is a trigonometric integral, obviously!

Recall that $\frac{d(\tan\theta)}{d\theta} = \sec^2\theta$. It seems wise to mention this since the power of the secant function is even and we have the identity $\tan^2\theta + 1 = \sec^2\theta$.

So,

$$\int \tan^2\theta \sec^4\theta\, d\theta = \int \tan^2\theta \sec^2\theta \sec^2\theta\, d\theta$$

$$= \int \tan^2\theta (\tan^2\theta + 1) \sec^2\theta\, d\theta$$

Let $u = \tan\theta$, then $du = \sec^2\theta\, d\theta$.

Our integral has the form $\int u^2(u^2+1)\, du = \int u^4 + u^2\, du$.

Which has integral $\frac{u^5}{5} + \frac{u^3}{3} + C$.

Thus, we obtain $\frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3} + C$.

3. Evaluate the integral.

$$\int \sqrt{1-9x^2}dx$$

The integrand has a square root which we wish to eliminate by forming a perfect square underneath it.

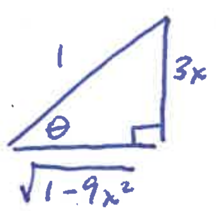
This radicand has the form of a difference of squares.

We will make a trigonometric substitution:

let $x = \frac{1}{3}\sin\theta$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ for then $1 - 9x^2 = 1 - 9\frac{1}{9}\sin^2\theta$

$$= 1 - \sin^2\theta$$

$$= \cos^2\theta .$$




With this substitution we have $dx = \frac{1}{3}\cos\theta\, d\theta$

Thus, the integral becomes

$$\int \sqrt{\cos^2\theta} \cdot \frac{1}{3}\cos\theta\, d\theta$$

$= \frac{1}{3}\int \cos^2\theta\, d\theta$ and by a ½∡ identity

$$= \frac{1}{3}\int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{3}\left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right] + C$$

$$\overset{*}{=} \frac{1}{6}\theta + \frac{1}{12} 2\sin\theta\cos\theta + C$$

$$= \frac{1}{6}\sin^{-1}(3x) + \frac{1}{6} 3x \cdot \sqrt{1-9x^2} + C$$

* using the double-angle formula

4. Evaluate the integral.

$$\int \frac{y}{(y+4)(2y-1)}\,dy$$

We will make a partial fraction decomposition.

Let $\dfrac{y}{(y+4)(2y-1)} = \dfrac{A}{y+4} + \dfrac{B}{2y-1}$

Then $y = A(2y-1) + B(y+4)$

If $y=-4$, then we obtain $-4 = A(-9) + 0$

$\Rightarrow A = 4/9$

If $y = \frac{1}{2}$, then we obtain $\frac{1}{2} = 0 + B(9/2)$

$\Rightarrow B = 1/9$

Thus the integral equals

$$\int \frac{4/9}{y+4}\,dy + \int \frac{1/9}{2y-1}\,dy$$

$$= \frac{4}{9}\ln|y+4| + \frac{1}{9}\cdot\frac{1}{2}\ln|2y-1| + C$$

5. [5 points each]

(a) In one sentence, explain why the integral $\int_0^\pi \tan x \, dx$ is improper. Then express (but do not evaluate) it as a sum of two improper integrals.

The integrand $\tan x$ is not continuous on the interval of integration $[0, \pi]$. Thus making it improper.

So, $\int_0^{\pi} \tan x \, dx = \int_0^{\pi/2} \tan x \, dx + \int_{\pi/2}^{\pi} \tan x \, dx$

$= \lim_{b \to \frac{\pi}{2}^-} \int_0^b \tan x \, dx + \lim_{b \to \pi/2^+} \int_b^{\pi} \tan x \, dx$

(b) Give an example of two functions $f(x)$ and $g(x)$ which illustrates that the following claim is false. *False claim: If $f(x) \leq g(x)$ and $\int_1^\infty g(x)dx$ diverges, then $\int_1^\infty f(x)dx$ also diverges.*

Here's one example of infinitely many possibilities

Let $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$. Then $\frac{1}{x^2} \leq \frac{1}{x}$

since $x^2 \geq x$ on $[1, \infty)$.

Both of these are "p-integrals" then.

$\int_1^\infty \frac{1}{x} dx$ diverges but $\int_1^\infty \frac{1}{x^2} dx$ converges.

6. Determine if the given integral is convergent or divergent.

$$\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}dx$$

Intuition: the degree of the numerator is 2 and the degree of the denominator is effectively $3 \cdot \frac{1}{2} = \frac{3}{2}$. Thus, the quotient reminds me of $x^{1/2}$ since $2 - 3/2 = 1/2$. However $\int_1^\infty x^{1/2}\,dx$ diverges since $x^{1/2} \to \infty$ as $x \to \infty$.

More precisely,

$$\frac{x^2}{\sqrt{1+x^3}} \approx \frac{x^2}{\sqrt{x^3}} = x^{1/2}$$

and $\int_1^\infty x^{1/2}$ diverges

OR

The integral equals $\lim_{b\to\infty} \int_0^b \frac{x^2}{\sqrt{1+x^3}}\,dx$

Make a u-substitution $u = 1+x^3$, $du = 3x^2dx$

So, we obtain

$$\lim_{b\to\infty} \frac{1}{3}\int_0^b \frac{3x^2\,dx}{\sqrt{1+x^3}} = \lim_{b\to\infty} \frac{1}{3}\cdot 2(1+x^3)^{1/2}\Big|_0^b$$

$$= \infty - \frac{2}{3} = \infty$$

# Trigonometric Identities

## Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

## Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

## Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

## Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$