Calculus II - Exam 1 - Fall 2022

October 6, 2022

Name: Solution Key

Honor Code Statement: I have neither given nor received unauthorized aid.

50 points total

Avg: $\frac{37.5}{50}$

S.D. 8.1 pts.

Directions: Complete all problems. Justify all answers/solutions. Calculators, texts or notes are not permitted. The value of each problem is indicated in brackets. Please remember the writing expectations that we've discussed in class while keeping the time-constraint in mind.

1. [8 points] Derive a formula for the derivative of $y = \tan^{-1}(x)$. Justify all your steps.

There are two possible approaches:

(1) As the ~~the~~ tangent function is a one-to-one differentiable function on $(-\pi/2, \pi/2)$, then by Theorem 7 of Section 6.1 its inverse function is differentiable on this interval and given by $y' = \dfrac{1}{\sec^2(\tan^{-1}x)}$. As $1+\tan^2 x = \sec^2 x$, we can write this as $y' = \dfrac{1}{1+\tan^2(\tan^{-1}x)} = \dfrac{1}{1+x^2}$.

OR

(2) We may use implicit differentiation

If $y = \tan^{-1}(x)$, then $\tan y = x$ and we may differentiate with respect to x. We obtain $\sec^2 y \cdot \dfrac{dy}{dx} = 1$

So, $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \dfrac{1}{\sec^2(\tan^{-1}x)}$. As above, this yields $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

2. [8 points] Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage V across its terminal and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

Given an equation for V in terms of t, using V_0 to denote the value of V when $t = 0$. How long will it take the voltage to drop to 10 percent of its original value? As calculators are not permitted, you will not need to express your answer in decimal notation.

We have already seen that the given equation is solved as

$$V = V_0 e^{kt}, \quad \text{where } k = -1/40$$

To answer the question: set $V = 0.1 V_0$ and solve for t.

$$0.1 V_0 = V_0 e^{-t/40}$$

$$0.1 = e^{-t/40}$$

$$\ln(0.1) = -t/40$$

$$-40 \ln(0.1) = t$$

Thus $t = -40 \ln(0.1)$ seconds

3. [3 points each] Find the derivative with respect to x of each of the following. State what rules you used.

(a) $y = \ln(\ln(x))$ We use the fact that $\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$ and the chain rule.

$$\frac{dy}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x}$$

(b) $y = e^{3x}x^{3e}$

We use the product rule and that $\frac{d(e^u)}{dx} = e^u \frac{du}{dx}$

$$y' = e^{3x} \cdot (3e)x^{3e-1} + 3e^{3x} \cdot x^{3e}$$

(c) $y = \log_2(1 - 3x)$

We use first a change of base formula $y = \frac{\ln(1-3x)}{\ln 2}$. Then $y' = \frac{1}{\ln 2} \cdot \frac{1}{1-3x} \cdot -3$ by a Chain Rule.

(d) $y = \int_2^{\sin(x)} \sin(t)dt$

Here we use FTC 1, and the Chain Rule.

$$y' = \sin(\sin x) \cdot \cos x$$

4. [3 points each] Find the following antiderivatives.

(a) $\int \frac{\sin(x)}{3+\cos(x)} dx$

We make a u-substitution. Let $u = 3 + \cos x$, then $du = -\sin x\, dx$

So the integral is

$$-\int \frac{-\sin x\, dx}{3+\cos x} = -\int \frac{du}{u} = -\ln|u| + C = -\ln|3+\cos x| + C.$$

(b) $\int x^2 e^{x^3} dx$

We make a u-substitution w/ $u = x^3$ and $du = 3x^2 dx$.

The integral is

$$\frac{1}{3}\int 3x^2 e^{x^3} dx = \frac{1}{3}\int e^u du = \frac{1}{3} e^u + C$$

$$= \frac{1}{3} e^{x^3} + C$$

5. [10 points]The following proof shows that the number e (also known as Euler's constant) can be thought of as a limit. To show that this limit does indeed equal e, we must do some manipulations. Along the way there are several indeterminate forms that are obtained. At each of the equations marked with a star $\star$ state the corresponding indeterminate form. At the equation marked with a double star $\star\star$, fill in the numerical answer.

Let

$$y =^{\star} \lim_{n\to\infty} (1+\frac{1}{n})^n. \qquad 1^{\infty}$$

If we take the natural logarithm of both sides, we get

$$\ln(y) = \ln(\lim_{n\to\infty} (1+\frac{1}{n})^n).$$

We may interchange as follows

$$\ln(y) =^{\star} \lim_{n\to\infty} \ln(1+\frac{1}{n})^n. \qquad 1^{\infty}$$

By a property of the natural logarithm, this equals

$$\ln(y) =^{\star} \lim_{n\to\infty} n\ln(1+\frac{1}{n}). \qquad \infty \cdot 0$$

Rewriting the product within the limit sign as a quotient, we obtain

$$\ln(y) =^{\star} \lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}}. \qquad \frac{0}{0}$$

To this, we may apply L'Hopital's Rule to obtain

$$\ln(y) = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\frac{-1}{n^2}}{\frac{-1}{n^2}}.$$

Simplifying, we obtain

$$\ln(y) = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}.$$

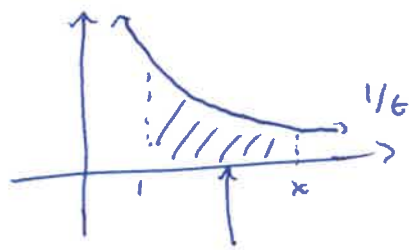
So, $\ln(y) =^{\star\star}$ 1 and so $y = e$.

6. [6 points] Give the definition of the natural logarithm function in symbols, words and a picture. Then use the same picture to define the number e and use words to define it. (This definition of e will necessarily be different to the one given in the previous problem.)

The natural logarithm of $x > 0$ corresponds to the area under the curve ~~of~~ $1/t$ between 1 and x.

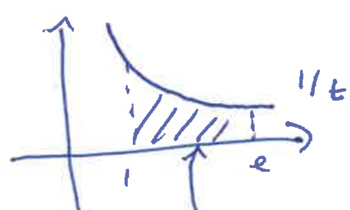
In symbols: $\displaystyle\int_1^x \frac{1}{t}\,dt = \ln(x)$, $x > 0$

In picture:



this area is $\ln(x)$.

The number e corresponds to when this area equals 1



an area of 1.