MULTIVARIABLE CALCULUS
EXAM 3
SPRING 2024

Name:
Honor Code Statement:

Directions: Each problem is worth 10 points and you may omit one of these (by putting a slash through it). Justify all answers/solutions. Electronic devices, books, and notes are not permitted. Please turn off cell phones and other devices; these should not be used under any circumstances. The last two pages contains formulas. Best of luck.

(1) [10 points] Use Fubini's Theorem to evaluate the double integral

$$\iint_R (x - 3y^2)\ dA$$

where $R = \{(x, y) | 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

Date: May 16, 2024.

(2) [10 points] **Sketch** the region D in the plane bounded by $y = x^2$ and $y = x^3$. Then **compute** this area by computing the double integral $\iint_D 1 dA$.

(3) [10 points] Rewrite the given sum of iterated integrals as a single iterated integral by reversing the order of integration. Sketch the region of integration. You do NOT need to evaluate the new integral obtained.

$$\int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy \; dy \; dx + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy \; dy \; dx$$

(4) Rewrite the given triple integral as an iterated integral where the order of integration is first z, then y and finally x. Sketch the region of integration. Sketch a projection of the region onto the x, y-plane.

$$\iiint_E \sqrt{x^2 + z^2}\; dV$$

where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. **DO NOT** evaluate.

(5) Evaluate $\iint_R (x)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. To do this, use a change to polar coordinates.

(6) [10 points] Calculate the following scalar line integral: $\int_C 2x\ ds$, where C is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$. Begin by parameterizing the curve.

(7) [10 points] Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{x}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, $0 \leq t \leq \frac{\pi}{2}$.

(8) [10 points]. The area of a region in the plane can be computed as $\iint_D 1 \, dx \, dy$. Green's Theorem allows us to compute this double integral as the following vector line integral

$$\oint_C -\frac{1}{2}y \, dx + \frac{1}{2}x \, dy.$$

Let us find the area inside the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

First, we must parameterize this curve. There are various parameterizations possible. **Which** of the following ones is suitable for this application and why?

- $\mathbf{x}_1(t) = (3\cos(t), -2\sin(t))$ where $0 \le t \le 2\pi$.
- $\mathbf{x}_2(t) = (3\cos(t), 2\sin(t))$ where $-\pi \le t \le \pi$

Now **find the area.**

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$

Change of variables in triple integrals:

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w))|\frac{\partial(x,y,z)}{\partial(u,v,w)}|dudvdw$$

Volume elements:

$$dV = dx\ dy\ dz\ Cartesian$$
$$dV = r\ dr\ d\theta\ dz\ Cylindrical$$
$$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\ spherical$$
$$dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\ dv\ dw\ general$$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Pythagorean and reciprocal identities

- If you don't know these, then get a tattoo.