MULTIVARIABLE CALCULUS
EXAM 3
SPRING 2024

Name:
Honor Code Statement:

Directions: Each problem is worth 10 points and you may omit one of these (by putting a slash through it). Justify all answers/solutions. Electronic devices, books, and notes are not permitted. Please turn off cell phones and other devices; these should not be used under any circumstances. The last two pages contains formulas. Best of luck.

(1) [10 points] Use Fubini's Theorem to evaluate the double integral

$$\iint_R (x - 3y^2)\, dA$$

where $R = \{(x, y) | 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

Fubini's Theorem tells us that we can evaluate this as an iterated integral:

$$\int_0^2 \int_1^2 x - 3y^2 \, dy \, dx = \int_0^2 xy - y^3 \Big|_1^2 \, dx$$

$$= \int_0^2 (2x - 8) - (x - 1) \, dx$$

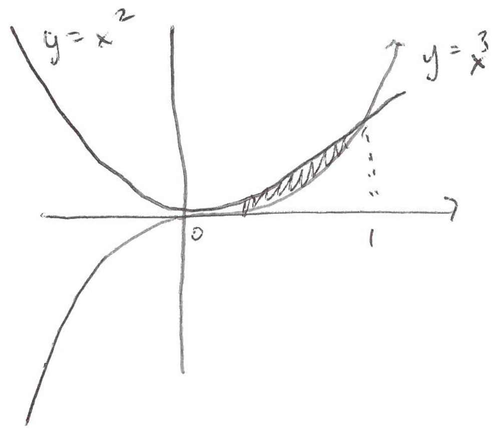
$$= \int_0^2 x - 7 \, dx$$

$$= \frac{x^2}{2} - 7x \Big|_0^2 = (2 - 14) - 0$$

$$= -12$$

Date: May 16, 2024.

(2) [10 points] **Sketch** the region in the plane bounded by $y = x^2$ and $y = x^3$. Then **compute** this area by computing the double integral $\iint_D 1 dA$.



$$\iint_D 1 \, dA$$

$$= \int_0^1 \int_{x^3}^{x^2} 1 \, dy \, dx$$

$$= \int_0^1 y \Big|_{x^3}^{x^2} dx$$

$$= \int_0^1 x^2 - x^3 \, dx$$

$$= \frac{x^3}{3} - \frac{x^4}{4} \Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

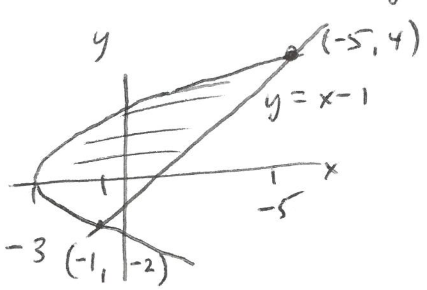
(3) [10 points] Rewrite the given sum of iterated integrals as a single iterated integral by reversing the order of integration. Sketch the region of integration. You do NOT need to evaluate the new integral obtained. ~~Indicate the advantage of reversing the order of integration — be specific.~~

$$\int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\ dy\ dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\ dy\ dx$$

Consider the bounds from the first portion:

$x=-3$ to $x=-1$ and $y=-\sqrt{2x+6}$ to $y=\sqrt{2x+6}$

These give $\frac{y^2-6}{2}=x \Rightarrow \frac{1}{2}y^2-3=x$



And the second portion is

$x=-1$ to $x=5$

and $y=x-1$ to $y=\sqrt{2x+6}$

gives $\frac{1}{2}y^2-3=x$

The line $x-1=y$ and the parabola $x=\frac{1}{2}y^2-3$ intersect

when $x=\frac{1}{2}(x-1)^2-3 \Rightarrow \frac{1}{2}x^2-x+\frac{1}{2}-3-x=0$

$\Rightarrow \frac{1}{2}x^2-2x-\frac{5}{2}=0 \Rightarrow x^2-4x-5=0$

$(x+1)(x-5)=0$

$\Rightarrow x=-1,\ x=5$

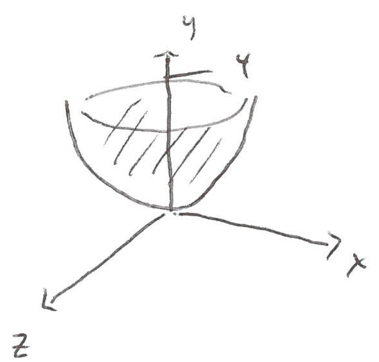
Thus,

$$\int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\ dx\ dy$$

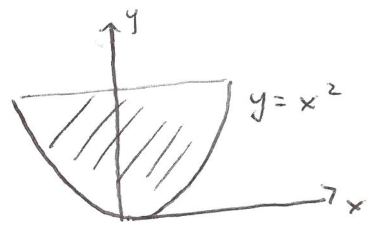
(4) Rewrite the given triple integral as an iterated integral where the order of integration is first z, then y and finally x. Sketch the region of integration. Sketch a projection of the region onto the x, y-plane.

$$\iiint_E \sqrt{x^2+z^2}\, dV$$

where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. Do not evaluate.



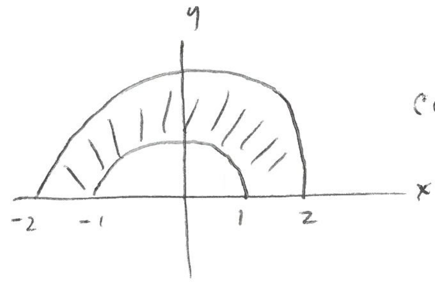
projection onto x,y-plane



$$\int_{x=-2}^{2}\int_{y=x^2}^{y=4}\int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy\, dx$$

(5) Evaluate $\iint_R (x)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. To do this, use a change to polar coordinates.

The region is



Concentric semi-circles.

Let $x = r\cos\theta$
$y = r\sin\theta$ and $x^2+y^2 = r^2$

The region is described as $0 \le \theta \le \pi$ and $1 \le r \le 2$.

Thus

$$\iint_R x\, dA = \int_0^{\pi}\int_1^2 r\cos\theta \cdot r\, dr\, d\theta$$

an "extra r" for the Jacobian

$$= \int_0^{\pi} \frac{r^3}{3}\cos\theta \Big|_1^2 \, d\theta$$

$$= \int_0^{\pi} \frac{7}{3}\cos\theta\, d\theta$$

$$= \frac{7}{3}\sin\theta \Big|_0^{\pi} = 0$$

Ah, yes, of course it is zero! The region is symmetric over the y-axis and the integrand is an odd function.

(6) [10 points] Calculate the following scalar line integral: $\int_C 2x\, ds$, where C is the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$. Begin by parametrizing the curve.

Let $x = t$ and then $y = t^2$.

So $x(t) = (t, t^2)$ $0 \le t \le 1$.

Then $\int_C 2x\, ds = \int_a^b f(\vec{x}(t))\, ||\vec{x}\,'(t)||\, dt$ where $f = 2x$

So, we obtain $x'(t) = (1, 2t)$ and $||\vec{x}\,'(t)|| = \sqrt{1+4t^2}$

$$\int_0^1 2t \sqrt{1+4t^2}\, dt$$

Let $u = 1+4t^2$. Then $du = 8t\, dt$

So the integral equals

$$\frac{1}{4}\int_0^1 8t\sqrt{1+4t^2}\, dt = \frac{1}{4}\cdot\frac{2}{3}\left(1+4t^2\right)^{3/2}\Big|_0^1$$

$$= \frac{1}{6}(5)^{3/2} - \frac{1}{6}\cdot(1+0)$$

$$= \frac{5^{3/2}-1}{6}$$

$\vec{x}$

(7) [10 points] Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, $0 \leq t \leq \frac{\pi}{2}$.

The work done is $\displaystyle\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\,dt$

We obtain

$$\int_0^{\pi/2} (\cos^2 t, -\cos t \sin t) \cdot (-\sin t, \cos t)\,dt$$

$$= \int_0^{\pi/2} -\sin t \cos^2 t - \sin t \cos^2 t \quad dt$$

$$= -2 \int_0^{\pi/2} \sin t \cos^2 t \, dt$$

Let $u = \cos t$

$du = -\sin t \, dt$

$$= +\frac{2}{3} \cos^3 t \Big|_0^{\pi/2}$$

$$= +\frac{2}{3}(0) - \frac{2}{3}(1) = -\frac{2}{3}$$

(8) [10 points]. The area of a region in the plane can be computed as $\iint_D 1\, dx\, dy$. Green's Theorem allows us to compute this double integral as the following vector line integral

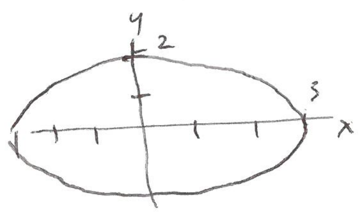
$$\oint_C -\frac{1}{2}y\, dx + \frac{1}{2}x\, dy.$$

Let us find the area inside the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

First, we must parameterize this curve. There are various parameterizations possible. Which of the following ones is suitable for this application and why?

- $\mathbf{x}(t) = (3\cos(t), -2\sin(t))$ where $0 \le t \le 2\pi$.
- $\mathbf{x}(t) = (3\cos(t), 2\sin(t))$ where $-\pi \le t \le \pi$

Now find the area.



The second parameterization keeps the region on the left, which is what Green's Theorem stipulates.

$$\oint_C -\frac{1}{2}y\, dx + \frac{1}{2}x\, dy = \int_{-\pi}^{\pi} -\frac{1}{2}\cdot 2\sin t \cdot (-3\sin t) + \frac{1}{2}\cdot 3\cos t \cdot 2\cos t\, dt$$

$$= \int_{-\pi}^{\pi} 3\sin^2 t + 3\cos^2 t\, dt$$

$$= \int_{-\pi}^{\pi} 3\, dt = 3t\Big|_{-\pi}^{\pi} = 3\pi - (-3\pi) = 6\pi$$

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ \ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ \ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ \ y = \rho\sin\varphi\sin\theta,\ \ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2 + y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \ \theta = \theta,\ \ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \ \theta = \theta$$

Change of variables in triple integrals:

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^*} f(x(u,v,w),y(u,v,w),z(u,v,w))|\frac{\partial(x,y,z)}{\partial(u,v,w)}|dudvdw$$

Volume elements:

$$dV = dx\ dy\ dz\ Cartesian$$
$$dV = r\ dr\ d\theta\ dz\ Cylindrical$$
$$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\ spherical$$
$$dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\ dv\ dw\ general$$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Pythagorean and reciprocal identities

- If you don't know these, then get a tattoo.