MULTIVARIABLE CALCULUS
EXAM 2
FALL 2025

Name:
Honor Code Statement:

Directions: You may **OMIT** one problem – do so by putting a slash through it. Justify all answers/solutions. Each problem is worth 10 points. Calculators/notes/texts/cell-phones are not permitted – the only permitted item is a writing utensil. Page 288 of the text is photocopied at the end of the exam. The exam is proctored by permission of the Dean of the Faculty. Best of luck.

(1) Find the arc length function for the curve $\mathbf{x}(t) = (t, t^2 - \frac{1}{8}\ln(t))$ with initial starting point $(1,1)$. (Hint: there will be a perfect square underneath the radical sign.)

As told to us by Equation (4) on page 205 of the text, the arc length function is

$$s(t) = \int_a^t \|\vec{x}\,'(\tau)\|\, d\tau$$

First we find $\|\vec{x}\,'(\tau)\| = \sqrt{1 + (2\tau - \frac{1}{8}\frac{1}{\tau})^2} = \sqrt{1 + 4\tau^2 - \frac{1}{2} + \frac{1}{64\tau^2}}$

$$= \sqrt{(2\tau + \frac{1}{8\tau})^2}$$

So,

$$s(t) = \int_1^t 2\tau + \frac{1}{8\tau}\, d\tau = \tau^2 + \frac{1}{8}\ln\tau \Big|_1^t$$

$$= (t^2 + \tfrac{1}{8}\ln t) - (1 + 0)$$

$$= t^2 + \tfrac{1}{8}\ln t - 1$$

Date: November 13, 2025.

(2) A moving particle starts at an initial position $\mathbf{x}(0) = (1, 0, 0)$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. **Find** its velocity $\mathbf{v}(t)$ at time t. **Find** its position $\mathbf{x}(t)$ at time t. **Write one sentence** that describes the position of the particle as t approaches infinity.

The acceleration is the derivative of the velocity. Thus,

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int 4t\vec{i} + 6t\vec{j} + 1\vec{k}\,dt$$

$$= 2t^2\vec{i} + 3t^2\vec{j} + t\vec{k} + \vec{C}$$

To determine the constant vector $\vec{C}$, we use that $\vec{v}(0) = \vec{i} - \vec{j} + \vec{k}$. Thus, $\vec{C} = \vec{v}(0)$

Thus, $\vec{v}(t) = (2t^2+1)\vec{i} + (3t^2-1)\vec{j} + (t+1)\vec{k}$

Now position is the antiderivative of velocity. So,

$$\vec{x}(t) = \int \vec{v}(t)\,dt = \int (2t^2+1)\vec{i} + (3t^2-1)\vec{j} + (t+1)\vec{k}$$

$$= \left(\frac{2t^3}{3} + t\right)\vec{i} + \left(t^3 - t\right)\vec{j} + \left(\frac{t^2}{2} + t\right)\vec{k} + \vec{C}_2$$

To determine $\vec{C}_2$ we use that $\vec{x}(0) = (1,0,0)$.

So $\vec{C}_2 = \vec{i}$

Thus, $\vec{x}(t) = \left(\frac{2t^3}{3} + t + 1\right)\vec{i} + (t^3 - t)\vec{j} + \left(\frac{t^2}{2} + t\right)\vec{k}$

Then as $t \to \infty$, the particle remains in the first octant "escaping" to "infinity".

(3) Calculate the flow line $\mathbf{x}(t)$ of the vector field $\mathbf{F}(x, y, z) = 2\mathbf{i} - 3y\mathbf{j} + z^3\mathbf{k}$ at the point $\mathbf{x}(0) = (3, 5, 7)$.

By Definition 3.2 on page 225, the flow line of $\vec{F}$ satisfies $\vec{x}\,'(t) = \vec{F}(\vec{x}(t))$.

So, $\vec{x}\,'(t) = (2, -3y, z^3)$

Thus $\dfrac{dx}{dt} = 2$ $\quad \dfrac{dy}{dt} = -3y$ $\quad \dfrac{dz}{dt} = z^3$ $\Big\}$ each is separable diff. eq.

$\int dx = \int 2\,dt$

$x = 2t + C$

At $t = 0$, $x = 3$.

Thus, $x = 2t + 3$

$\int \dfrac{dy}{y} = \int -3\,dt$

$\ln|y| = -3t$

$y = e^{-3t + C}$

$y = Ce^{-3t}$

At $t = 0$, $y = 5$

thus $5 = Ce^0$ so $C = 5$

And, $y = 5e^{-3t}$

$\int \dfrac{dz}{z^3} = \int dt$

$\dfrac{z^{-2}}{-2} = t + c$

$z^2 = \dfrac{1}{-2(t+c)}$

$z = \sqrt{\dfrac{1}{-2(t+c)}}$

At $t = 0$, $z = 7$.

Thus, $7 = \sqrt{\dfrac{1}{-2c}}$

$\Rightarrow c = -\dfrac{1}{98}$

so $z = \sqrt{\dfrac{1}{-2(t - \frac{1}{98})}}$

$$\Rightarrow \vec{x}(t) = \left(2t+3,\ 5e^{-3t},\ \sqrt{\frac{1}{-2(t - \frac{1}{98})}}\right)$$

(4) For the same vector field as in the previous problem: calculate the divergence of **F** at this point, and calculate the curl of **F** at this point.

The divergence is found by computing

$$\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 0 - 3 + 3z^2 = 3z^2 - 3.$$

The curl is found by computing

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2 & -3y & z^3 \end{vmatrix}$$

$$= i\left(\frac{\partial z^3}{\partial y} - \frac{\partial(-3y)}{\partial z}\right) - \vec{j}\left(\frac{\partial z^3}{\partial x} - \frac{\partial z}{\partial z}\right) + \vec{k}\left(\frac{\partial(-3y)}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$= 0$$

The field is irrotational.

(5) Find the first- and second-order Taylor polynomials for the function

$$f(x, y) = ye^{3x}$$

at the point $\mathbf{a} = (0, 0)$. Use this second order Taylor polynomial to estimate $f(0.1, 0.1)$.

We first compute the needed 1st-order and 2nd-order partial derivatives.

$$\frac{\partial f}{\partial x} = 3ye^{3x} \qquad \frac{\partial^2 f}{\partial x^2} = 9ye^{3x} \qquad \frac{\partial^2 f}{\partial y\, \partial x} = 3e^{3x}$$

$$\frac{\partial f}{\partial y} = e^{3x} \qquad \frac{\partial^2 f}{\partial y^2} = 0 \qquad \frac{\partial^2 f}{\partial x\, \partial y} = 3e^{3x}$$

$$p_1(x,y) = f(0,0) + f_x(0,0)\,(x-0) + f_y(0,0)\,(y-0)$$

$$= 0 + 0\,(x-0) + 1\,(y-0) = y$$

See page 254-255

$$p_2(x,y) = p_1(x,y) + \tfrac{1}{2} f_{xx}(0,0)\,(x-0)^2 + f_{xy}(0,0)\,(x-0)(y-0) + \tfrac{1}{2} f_{yy}(0,0)(y-0)^2$$

$$= y + \tfrac{1}{2}\cdot 0 \cdot x^2 + 3(x-0)(y-0) + \tfrac{1}{2}\cdot 0 \cdot y^2$$

$$= y + 3xy$$

Now

$$f(0.1, 0.1) \approx p_2(0.1, 0.1) = 0.1 + 3(.1)(.1)$$

$$= 0.13$$

(6) Find the one critical point of the following function. Then use the second derivative test for functions of two variables to determine the nature of this critical point.

$$f(x, y, z) = x^2 - xy + z^2 - 2xz + 6z$$

To find the critical point, we set the partial derivatives to zero.

$$\left.\begin{array}{ll} ① & \dfrac{\partial f}{\partial x} = 2x - y - 2z \\ ② & \dfrac{\partial f}{\partial y} = -x \\ ③ & \dfrac{\partial f}{\partial z} = 2z - 2x + 6 \end{array}\right\} \Rightarrow$$

setting to zero we see $x = 0$ by ②

Plugging into ③ we see

$2z - 0 + 6 = 0 \Rightarrow z = -3$

Then plugging into ①, we see

$2 \cdot 0 - y - 2(-3) = 0 \Rightarrow y = 6$

Thus, the only critical point is $(0, 6, -3)$.

We construct the Hessian matrix to test its nature

$$H = \begin{bmatrix} 2 & -1 & -2 \\ -1 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

The sequence of partial minors is:

$d_1 = 2$, $d_2 = -1$, $d_3 = -2$.

Thus, the critical point is a saddle point.

(7) Find the points on the sphere $x^2+y^2+z^2 = 4$ that are closest to and farthest from the point $(1,1,1)$. Do so using the method of Lagrange multipliers. (Hint: the function you wish to optimize is the distance between a given point (x,y,z) and the point $(1,1,1)$. However, optimizing the square of this distance is preferable. Why?)

The distance from (x,y,z) to $(1,1,1)$ is

$$d = \sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}$$

The point that minimizes this distance will also minimize the distance squared. So $d^2 = (x-1)^2 + (y-1)^2 + (z-1)^2$.

Let $g(x,y,z) = x^2+y^2+z^2$. So $g(x,y,z) = 4$ is our constraint.

By Lagrange's method we consider $\nabla f(\vec{x}) = \lambda \nabla g(\vec{x})$ and $g = 4$:

$$2(x-1)\ \cancel{2x} = \lambda 2x$$
$$2(y-1)\ \cancel{2y} = \lambda 2y$$
$$2(z-1)\ \cancel{2z} = \lambda 2z$$
$$x^2+y^2+z^2 = 4$$

From ①, ~~$2x - 2\lambda x = 0 \Rightarrow x(1-\lambda$~~

$$2x - 2 = \lambda 2x \Rightarrow x - \lambda x = 1 \Rightarrow x = \frac{1}{1-\lambda}$$

Similarly $y = \frac{1}{1-\lambda}$ and $z = \frac{1}{1-\lambda}$

Thus,

$$\frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 4$$

$$\frac{3}{(1-\lambda)^2} = 4 \Rightarrow \cancel{\tfrac{4}{3}}\ \frac{3}{4} = (1-\lambda)^2$$

$$\pm\sqrt{3/4} = 1-\lambda \qquad \lambda = 1 \pm \sqrt{3/4}$$

(7) Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(1,1,1)$. Do so using the method of Lagrange multipliers. (Hint: the function you wish to optimize is the distance between a given point (x,y,z) and the point $(1,1,1)$. However, optimizing the square of this distance is preferable. Why?) Avoid using a technical argument for determining which is closest and which is farthest and use some geometric reasoning/intuition instead.

Continued

As $\lambda = 1 \pm \frac{\sqrt{3}}{2}$ we get $x = \frac{1}{1-\left(1 \pm \frac{\sqrt{3}}{2}\right)}$

$$x = \frac{1}{\mp \frac{\sqrt{3}}{2}} = \mp \frac{2}{\sqrt{3}}$$

likewise for y and z.

Thus, the critical points are $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

and $\left(\frac{-2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$.

As these two points are collinear with $(1,1,1)$, it is easy to see that the first one is closest and the second one farthest.