Total points: 70

Average 59.6

MULTIVARIABLE CALCULUS
EXAM 1
FALL 2025

Name:
Honor Code Statement:

Directions: Each question or one of its subparts is worth 5 points – you may omit any one of these by putting a slash through it. Justify all answers/solutions. No outside materials are permitted. A formula sheet is on the last page. The exam is proctored by permission of the Dean of the Faculty. Best of luck!

(1) [5 points] Evaluate this limit by making a change of variables to polar coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2}$$

We make use of the following:

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$.

As $(x,y) \to (0,0)$, $r \to 0$. Thus, the limit becomes

$$\lim_{r\to 0} \frac{r^2\cos^2\theta + r^2\cos\theta\sin\theta + r^2\sin^2\theta}{r^2} = \lim_{r\to 0} 1 + \cos\theta\sin\theta.$$

So, we see that the limit depends upon the angle of approach. Thus, the limit does not exist

Date: October 8, 2025.

(2) [5 points] Calculate the directional derivative of the function $f(x, y) = \frac{1}{x^2+y^2}$ at $\mathbf{a} = (3, -2)$ in the direction parallel to the vector $\mathbf{u} = \mathbf{i} - \mathbf{j}$.

By Theorem 6.2, the directional derivative may be calculated as the dot product of the gradient vector and the direction as a unit vector.

So, $\nabla f(x,y) = \left(\frac{-2x}{(x^2+y^2)^2}, \frac{-2y}{(x^2+y^2)^2} \right) \Rightarrow \nabla f(3,-2) = \left(\frac{-6}{169}, \frac{4}{169} \right)$

Normalizing $\vec{u}$ we get $\hat{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$

So, $D_{\vec{u}} f(\vec{a}) = \left(\frac{-6}{169}, \frac{4}{169} \right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) = \left[\frac{-6}{169\sqrt{2}} + \frac{-4}{169\sqrt{2}} \right]$

$= \frac{-10}{169\sqrt{2}}$

(3) [5 points] Find the matrix of partial derivatives of the function $\mathbf{f}(x, y) = (x^2, xy, \sin(x))$. Use Theorem 3.10 to justify that the function is differentiable at the point $(1, 1)$.

$$Df(x,y) = \begin{bmatrix} 2x & 0 \\ y & x \\ \cos x & 0 \end{bmatrix}$$

Since all these partial derivatives are continuous everywhere, the function is differentiable everywhere.

Thus,

$$Df(1,1) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ \cos(1) & 0 \end{bmatrix}$$

(d) Find the distance between the plane found in the first part and the point $(0, 0, 0)$.

(See Example 8 in Section 1.5.)

The point $P_1 = (1, 2, 3)$ is in the first plane, and $P_4 = (0,0,0)$ in the second. So, $\overrightarrow{P_1P_4} = (-1, -2, -3)$. We find the projection of this vector onto the normal vector, and then find the length of this vector.

$$\text{proj}_{\vec{n}} \overrightarrow{P_1P_4} = \left(\frac{\vec{n} \cdot \overrightarrow{P_1P_4}}{\vec{n} \cdot \vec{n}} \right) \vec{n} = \frac{-35}{154} \vec{n}$$

Thus, the distance is $\left\| \frac{-35}{154} \vec{n} \right\| = \frac{35}{154} \sqrt{154}$

(e) Find the parametric equations for the plane found in the first part of this question.

We use Proposition 5.1 of Section 1.5

$$\vec{x}(s,t) = s\overrightarrow{P_1P_2} + t\,\overrightarrow{P_1P_3} + \overrightarrow{OP_1}$$
$$= s \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

thus

$$x = 3s + 0t + 1$$
$$y = -2s + 1t + 2$$
$$z = -2s - 3t + 3$$

(4) [25 points]

(a) Find the equation of the plane that contains the following three points: $(1, 2, 3), (4, 0, 1)$ and $(1, 3, 0)$.

Let $P_1 = (1,2,3)$, $P_2 = (4,0,1)$, $P_3 = (1,3,0)$. We find two vectors in the plane and from this a normal vector to the plane.

$$\overrightarrow{P_1P_2} = (3,-2,-2) \quad , \overrightarrow{P_1P_3} = (0,1,-3)$$

The normal vector is $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & -2 \\ 0 & 1 & -3 \end{vmatrix} = 8\hat{i} + 9\hat{j} + 3\hat{k} = \begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$.

Thus, by Equation 2 of Chapter 1.5:

$$8(x-1) + 9(y-2) + 3(z-3) = 0$$

$$8x + 9y + 3z = 35$$

(b) Give a vector that is normal to the plane.

$\vec{n} = \begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$ as shown above.

(c) Find the equation for a plane parallel to the one you found that contains the point $(0, 0, 0)$.

The plane parallel has the same normal vector but contains the point $(0,0,0)$:

$$8(x-0) + 9(y-0) + 3(z-0) = 0$$

$$8x + 9y + 3z = 0.$$

(e) Find the gradient vector at the point $(1, 1)$.

$$\nabla f(x,y) = (8x, 18y)$$

$$\nabla f(1,1) = (8, 18)$$

(f) Use this information to find the normal line to the level curve C associated with the point $(1, 1)$ at this point. That is, find the line perpendicular to C.

By Theorem 6.4 of Section 2.6, the gradient vector is perpendicular to the level curve. This gives the direction to the line, and we are given a point on the line.

$$\vec{\ell} = (1,1) + t(8,18) \quad \text{or} \quad \begin{cases} x = 1 + 8t \\ y = 1 + 18t \end{cases}$$

(g) Find the equation of the tangent plane at the point $(1, 1, 13)$.

The equation for the tangent hyperplane at $(\vec{a}, f(\vec{a}))$ is given by

$$h(x,y) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

see Equation 8 in Section 2.3

Thus,

$$h(x,y) = 13 + (8,18) \cdot (x-1, y-1)$$

$$= 13 + 8(x-1) + 18(y-1)$$

$$= 8x + 18y - 13$$

(5) [35 points] Consider the following function with codomain $\mathbb{R}$: $f(x,y) = 4x^2 + 9y^2$.

(a) State the domain and range of the function. Is the function onto? Why or why not?

The domain is $\mathbb{R}^2$. The range is all non-negative reals. The function is <u>not</u> onto because there are points in the codomain w/ no preimage, e.g. $c = -12$.

(b) Provide a counterexample to the statement, "f is one-to-one."

f is many to one since $f(1,0) = 4$ and $f(-1,0) = 4$

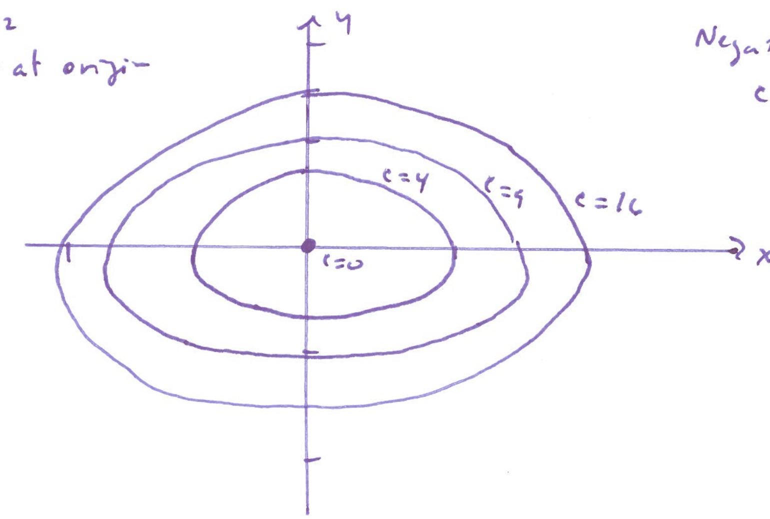
(c) Determine/draw several level curves of the given function. Do so for the values $c = 0, c = 4, c = 9$ and one other value of c.

$c = 0 \Rightarrow 0 = 4x^2 + 9y^2$
a point at origin

$c = 4 \Rightarrow 4 = 4x^2 + 9y^2$
$1 = x^2 + \frac{y^2}{4/9}$
an ellipse

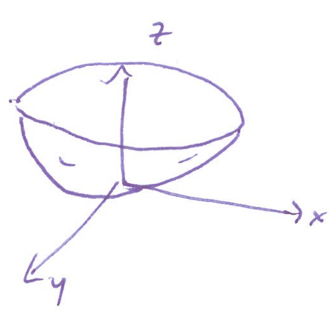
$c = 9 \Rightarrow 9 = 4x^2 + 9y^2$
$1 = \frac{x^2}{9/4} + y^2$
an ellipse

$\vdots$



Negative values of c do not produce a level curve.

(d) Use this information to sketch or describe the graph of f.



It is an upward facing elliptical bowl.

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2 + y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$