Isospectral Metrics on Classical Compact Simple Lie Groups

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Abstract

We prove the existence of nontrivial multiparameter isospectral deformations of metrics on the classical compact simple Lie groups $SO(n)$ ($n = 9, n \geq 11$), $\mathrm{Spin}(n)$ ($n = 9, n \geq 11$), $SU(n)$ ($n \geq 7$), and $Sp(n)$ ($n \geq 5$). The proof breaks into three main steps. First we outline a method devised by Schueth for constructing metrics on these groups from linear maps. Isospectrality and equivalence of linear maps are defined and we invoke a theorem by Schueth which states that isospectral linear maps give rise to isospectral metrics. Next we prove the existence of multidimensional families of linear maps such that within each family the maps are pairwise isospectral and pairwise not equivalent. Finally, we prove that generically, if $\mathcal{F}$ is a family of metrics arising from a collection of pairwise nonequivalent linear maps, then for any metric g contained in $\mathcal{F}$ there are at most finitely many metrics in $\mathcal{F}$ which are isometric to g. Thus we conclude the existence of nontrivial multiparameter isospectral deformations of metrics on the classical compact simple Lie groups of large enough dimension.

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I dedicate this thesis to my parents. Thank you for loving me through every twist and turn of graduate school. I love you more than I can say.

Contents

Chapter 1

Introduction

Spectral geometry has its roots in spectroscopy and the problem of determining the chemical composition of stars. By examining the natural vibrational frequencies, or the spectrum, of a star, astronomers are able to deduce the chemical composition of the star. More specifically, spectral geometry is the branch of differential geometry concerned with examining the interplay of the spectrum of a compact Riemannian manifold M with the geometry and topology of M. For example, if one knows the entire spectrum of M, then one also knows the volume, dimension, and total scalar curvature of M. Observations of this nature indicate that the spectrum of a Riemannian manifold is intimately tied to its geometry and topology. However, in 1964, Milnor proved that the spectrum of M does not necessarily encode all geometric information about M by exhibiting a pair of 16-dimensional tori which are isospectral but not isometric. In this paper, we prove a result similar in nature. We combine ideas from Gordon and Wilson and from Schueth to produce multidimensional families of isospectral metrics on each of $SO(n)$, $\mathrm{Spin}(n)$, $SU(n)$, and $Sp(n)$. Note that these are all of the classical compact simple Lie groups. The appropriate choices of n depend on the group, but in all cases the result holds except for a few small values of n.

These are the first examples of multidimensional families of isospectral left-invariant metrics on a compact simple Lie group and the first examples of isospectral metrics of any form on $Sp(n)$.

The paper is organized as follows. In Chapter 2, we review the basic definitions and properties of Lie groups and Lie algebras. In particular, we discuss the correspondence between Lie groups and Lie algebras which allows us to translate questions about Lie groups into simpler questions about Lie algebras.

In the first half of Chapter 3, we state the fundamental definitions and results of spectral geometry. We then place our result into context by surveying several examples of isospectral manifolds, with a focus on manifolds having different local geometry.

We begin the examination of our particular problem in the second half of Chapter 3. Here we describe a method for constructing metrics on $SO(n+4)$, $\mathrm{Spin}(n+4)$, $SU(n+3)$, and $Sp(n+2)$ from linear maps $j : \mathfrak{h} \to \mathfrak{g}_n$ where $\mathfrak{h}$ denotes the Lie algebra of the two-dimensional torus and $\mathfrak{g}_n$ is one of $\mathfrak{so}(\mathrm{n})$, $\mathfrak{su}(\mathrm{n})$, or $\mathfrak{sp}(\mathrm{n})$. Our goal becomes to reformulate our geometric question about metrics into an algebraic question about linear maps. More specifically, we define two relationships between linear maps: isospectrality and equivalence. A theorem by Schueth tells us that isospectral maps give rise to isospectral metrics.

Chapter 4 is devoted entirely to the proof of

Theorem 4.1.1 *Let $\mathfrak{g}_n$ be one of $\mathfrak{so}(\mathrm{n})$ $(n = 5, n \geq 7)$, $\mathfrak{su}(\mathrm{n})$ $(n \geq 4)$, or $\mathfrak{sp}(\mathrm{n})$ $(n \geq 3)$. Let L be the space of all linear maps $j : \mathfrak{h} \to \mathfrak{g}_n$. There exists a Zariski open set $\mathcal{O} \subset L$ such that each $j_0 \in \mathcal{O}$ is contained in a d-parameter family of j-maps which are isospectral but not equivalent. Here d depends on $\mathfrak{g}_n$ as follows:*

$\mathfrak{g}_n$	d
$\mathfrak{so}(\mathrm{n})$	$d \geq n(n-1)/2 - [\frac{n}{2}]([\frac{n}{2}]+2)$
$\mathfrak{su}(\mathrm{n})$	$d \geq n^2 - 1 - \frac{n^2+3n}{2}$
$\mathfrak{sp}(\mathrm{n})$	$d \geq n^2 - n$

Theorem 4.1.1 was originally proven for $\mathfrak{so}(\mathrm{n})$ by Gordon and Wilson. We extend the proof to include $\mathfrak{su}(\mathrm{n})$ and $\mathfrak{sp}(\mathrm{n})$.

In Chapter 5, we complete our examination by proving a general nonisometry principle for families of metrics arising from the construction in Chapter 3. In particular we prove

Theorem 5.1.12 *Suppose $j_0 : \mathfrak{h} \to \mathfrak{g}_n$ is contained in a family of generic linear maps which are pairwise nonequivalent. For $\mathfrak{so}(\mathrm{n})$ ($n = 5, 7$ and $n \geq 9$), $\mathfrak{su}(\mathrm{n})$ ($n \geq 2$), and $\mathfrak{sp}(\mathrm{n})$ ($n \geq 3$), there is at most one other linear map j in the family such that g_{j_0} and g_j are isometric. For $\mathfrak{so}(8)$ there are at most five other maps.*

Thus the families of linear maps guaranteed by Theorem 4.1.1 lead us to continuous nontrivial d-parameter isospectral deformations of metrics on each of $SO(n)$ ($n = 9, n \geq 11$), Spin(n) ($n = 9, n \geq 11$), $SU(n)$ ($n \geq 7$), and $Sp(n)$, ($n \geq 5$). Except for the case of $SU(7)$, all deformations have dimension greater than 1.

Chapter 2

Lie Groups and Lie Algebras

The theory of Lie groups and Lie algebras is a richly developed and beautiful area of mathematics. In this chapter, we give the basic definitions and examples of Lie groups and Lie algebras. We also indicate the fundamental relationship between Lie groups and Lie algebras. We then state results and properties which we will need for later chapters. For a more full treatment of Lie groups and Lie algebras, the reader is encouraged to see [FH91], [Hel78], or [Kna96].

2.1 Introduction to Lie Groups

We begin with the main definitions.

A **Lie group** is a smooth manifold G endowed with a group structure such that the maps $\mu : G \times G \to G$, $\mu(x, y) = xy$ and $\iota : G \to G$, $\iota(x) = x^{-1}$ are smooth. A **Lie subgroup** H of a Lie group G is a subgroup with the structure of a Lie group such that the inclusion mapping is an immersion. If the inclusion mapping is an embedding, we say that H is a **closed Lie subgroup** of G. Given two Lie groups, G and G', a map $f : G \to G'$ is a **Lie group homomorphism** if it is a smooth group homomorphism.

The following examples of Lie groups will figure heavily in this paper.

Example 2.1.1. The n-dimensional torus, T^n.

The torus T^n is the direct product of n copies of S^1. Here we think of S^1 as the set of complex numbers of unit modulus, and multiplication of elements of S^1 is given by complex multiplication. Since multiplication on each component is given by complex multiplication, we see that T^n is an abelian Lie group.

Example 2.1.2. Matrix groups.

The most basic examples of matrix groups are $GL_{\mathbb{R}}(n)$ and $GL_{\mathbb{C}}(n)$, the groups of $n \times n$ invertible matrices with real and complex entries respectively. We may think of $GL_{\mathbb{R}}(n)$ (resp. $GL_{\mathbb{C}}(n)$) as the group of linear automorphism of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$).

We will be particularly concerned with the following subgroups of $GL_{\mathbb{R}}(n)$ and $GL_{\mathbb{C}}(n)$. Here we denote the $n \times n$ identity matrix by I_n, the transpose of a matrix g by g^t, the complex conjugate of g by $\overline{g}$, and the conjugate transpose of g by g^*.

- $O(n) \subset GL_{\mathbb{R}}(n)$.

  The subgroup of matrices g satisfying $g^t g = I_n$. $O(n)$ is the set of orthogonal invertible linears transformations of $\mathbb{R}^n$ with the standard inner product.

- $SO(n) \subset GL_{\mathbb{R}}(n)$.

  The subgroup of matrices g satisfying $\det g = 1$ and $g^t g = I_n$. $SO(n)$ is the set of orthogonal, orientation-preserving invertible linear transformations of $\mathbb{R}^n$.

- $SU(n) \subset GL_{\mathbb{C}}(n)$.

  The subgroup of matrices g satisfying $\det g = 1$ and $g^* g = I_n$. Thinking of $\mathbb{C}^n$ as row vectors of length n, if we define a Hermitian inner product H on

$\mathbb{C}^n$ by $H(v,w) = v^* \cdot w$, then $SU(n)$ is the subgroup of orientation-preserving invertible complex linear transformations of $\mathbb{C}^n$ which preserve H.

- $Sp(n) \subset GL_{\mathbb{C}}(2n)$.

  Let
  $$M = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \tag{2.1}$$
  $Sp(n)$ is the subgroup of matrices g satisfying $g^t M g = M$ and $g^* g = I_{2n}$. If we define a skew-symmetric bilinear form Q on $\mathbb{C}^{2n}$ by $Q(x,y) = {}^t x M y$, then $Sp(n)$ is the subgroup of invertible complex linear transformations of $\mathbb{C}^{2n}$ which preserve both the Hermitian inner product, H, defined above and Q.

All three of $SO(n)$, $SU(n)$, and $Sp(n)$ are compact and connected for $n \geq 1$. For $n \geq 1$, both $SU(n)$ and $Sp(n)$ are simply connected manifolds, however the fundamental group of $SO(n)$ has order 2 for $n \geq 3$.

Example 2.1.3. Spin(n).

For $n \geq 3$, Spin(n) is the universal cover of $SO(n)$. Since the fundamental group of $SO(n)$ is $\mathbb{Z}_2$, we have that Spin(n) is a two-fold covering of $SO(n)$.

Lie groups are particularly nice objects of study due to the strong relationship between the group structure and the manifold structure. In particular, we may restrict our attention to vector fields and metrics which are invariant under the group action.

For $g \in G$, denote left- (resp. right-) multiplication by g by L_g (resp. R_g). A vector field X on a Lie group G is **left-invariant** if $L_{g*}(X_h) = X_{gh}$ for all $g, h \in G$. Similarly, a metric μ on G is called **left-invariant** if $L_g^* \mu = \mu$ for all $g \in G$. We say μ is **bi-invariant** if $L_g^* \mu = R_g^* \mu = \mu$ for all $g \in G$.

All metrics which we consider in this paper are left-invariant. If G is a compact connected Lie group, there exists exactly one (up to scaling) bi-invariant metric on G.

Finally, we state two useful facts about isometry groups of manifolds. Suppose that M is any Riemannian manifold and let $I(M)$ be the group of isometries of M. A classical result of Myers and Steenrod says that $I(M)$ with the compact open topology is a Lie group [MS39]. Furthermore, if we suppose that M is compact, we may conclude that under the compact open topology, $I(M)$ is also compact.

2.2 Introduction to Lie Algebras

As we will see in the next section, Lie algebras are the linear counterparts to Lie groups. However, they may be defined independently of Lie groups.

A **Lie algebra** $\mathfrak{g}$ over a field $\mathfrak{k}$ is a vector space over $\mathfrak{k}$ together with a skew-symmetric product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the **Jacobi identity** $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$. A **Lie subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ such that $[X,Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, we say $\mathfrak{h}$ is an **ideal** of $\mathfrak{g}$. The **center** of a Lie algebra $\mathfrak{g}$ is the subalgebra of elements X satisfying $[X,Y] = 0$ for all $Y \in \mathfrak{g}$. We say that $\mathfrak{g}$ is **abelian** if $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$.

Example 2.2.1. Matrix algebras

The set $\mathfrak{gl}_{\mathbb{R}}(\mathrm{n})$ (resp. $\mathfrak{gl}_{\mathbb{C}}(\mathrm{n})$) of $n \times n$ matrices over $\mathbb{R}$ (resp. $\mathbb{C}$) is a Lie algebra with product given by $[X,Y] = XY - YX$. Here $\mathfrak{gl}_{\mathbb{R}}(\mathrm{n})$ is a real Lie algebra and we may consider $\mathfrak{gl}_{\mathbb{C}}(\mathrm{n})$ either a real or a complex Lie algebra. Subalgebras of $\mathfrak{gl}_{\mathbb{R}}(\mathrm{n})$ and $\mathfrak{gl}_{\mathbb{C}}(\mathrm{n})$ are known as matrix algebras. Matrix algebras are important examples of Lie algebras because in fact, according to Ado's theorem, any Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ can be realized as a matrix algebra.

The following are examples of real matrix algebras which will be of concern to us.

- $\mathfrak{so}(\mathrm{n}) \subset \mathfrak{gl}_{\mathbb{R}}(\mathrm{n})$. The set of real matrices which are skew symmetric ($A+A^t=0$). We see that the diagonal entries of any element of $\mathfrak{so}(\mathrm{n})$ are zero, so each element of $\mathfrak{so}(\mathrm{n})$ is traceless. The dimension of $\mathfrak{so}(\mathrm{n})$ is $\frac{n(n-1)}{2}$. Notice that for $A \in \mathfrak{so}(\mathrm{n})$, any odd power of A will also be an element of $\mathfrak{so}(\mathrm{n})$. This follows from induction on the defining identity $A = -A^t$.

- $\mathfrak{su}(\mathrm{n}) \subset \mathfrak{gl}_{\mathbb{C}}(\mathrm{n})$. The set of complex skew Hermitian matrices ($A + A^* = 0$) with trace zero. The real dimension of $\mathfrak{su}(\mathrm{n})$ is $n^2 - 1$.

- $\mathfrak{sp}(\mathrm{n}) \subset \mathfrak{gl}_{\mathbb{C}}(2\mathrm{n})$. The set of $n \times n$ skew Hermitian matrices satisfying $A^tM + MA = 0$ where

$$M = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \tag{2.2}$$

(as in the definition of $Sp(n)$).

Direct calculation shows that elements of $\mathfrak{sp}(\mathrm{n})$ are of the form

$$\begin{bmatrix} Z_{11} & Z_{12} \\ -\overline{Z}_{12} & \overline{Z}_{11} \end{bmatrix} \tag{2.3}$$

where Z_{11} is an $n \times n$ skew Hermitian matrix and Z_{12} is an $n \times n$ symmetric matrix. The real dimension of $\mathfrak{sp}(\mathrm{n})$ is $2n^2 + n$. The eigenvalues of elements of $\mathfrak{sp}(\mathrm{n})$ come in plus and minus pairs, so as with $\mathfrak{so}(\mathrm{n})$, we see that each element of $\mathfrak{sp}(\mathrm{n})$ is automatically traceless. We may use the identity $A^tM + MA = 0$ and induction to show that for $A \in \mathfrak{sp}(\mathrm{n})$, all odd powers of A are also elements of $\mathfrak{sp}$(n).

It is easy to check that each of $\mathfrak{so}$(n), $\mathfrak{su}$(n), and $\mathfrak{sp}$(n) consists of **normal** elements, that is, elements A such that $AA^* = A^*A$. Basic linear algebra then implies that each element of $\mathfrak{so}$(n), $\mathfrak{su}$(n), or $\mathfrak{sp}$(n) is diagonalizable by a unitary (or orthogonal in the case of $\mathfrak{so}$(n)) matrix.

2.3 The Fundamental Relationship

In this section we make precise the relationship between Lie groups and Lie algebras. We will make heavy use of this relationship in order to answer questions about Lie groups in the much simpler Lie algebra setting.

Let G be a Lie group and let $\mathfrak{g}$ be the vector space of left-invariant vector fields on G. Then $\mathfrak{g}$ is a Lie algebra where the bracket is given by $[X, Y] = XY - YX$. The map from $\mathfrak{g}$ to T_eG which sends a left-invariant vector field to its value at the identity is a vector space isomorphism. Thus we may identify $\mathfrak{g}$ with T_eG.

Thinking of $\mathfrak{g}$ as T_eG, suppose $X \in \mathfrak{g}$. Then there is a unique smooth homomorphism $\theta_X : \mathbb{R} \to G$ such that $\dot{\theta}_X(0) = X$. We define the exponential map $\exp : \mathfrak{g} \to G$ by

$$\exp X = \theta_X(1). \tag{2.4}$$

The exponential map is the tool which allows us to translate between a Lie group and its Lie algebra. In particular, it gives us a method for relating subalgebras of $\mathfrak{g}$ with subgroups of G.

Suppose H is a subgroup of G. The differential of the inclusion mapping from H into G gives an injective map from the Lie algebra $\mathfrak{h}$ of H into the Lie algebra $\mathfrak{g}$ of G. Thus $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. On the other hand, suppose $\mathfrak{h}$ is any subalgebra of

$\mathfrak{g}$. Then $\mathfrak{h}$ is the Lie algebra of exactly one connected subgroup H of G. In fact, H is the subgroup generated by the set $\{\exp X | X \in \mathfrak{h}\}$.

Example 2.3.1. We have the following correspondences between the Lie groups defined in Section 2.1 and the Lie algebras defined in Section 2.2:

- $\mathfrak{so}$(n) is the Lie algebra of $SO(n)$.
- $\mathfrak{su}$(n) is the Lie algebra of $SU(n)$.
- $\mathfrak{sp}$(n) is the Lie algebra of $Sp(n)$.
- $\mathfrak{so}$(n) is also the Lie algebra of Spin(n).

Under the correspondence between Lie subalgebras and Lie subgroups, certain properties are preserved. For example, normal subgroups of G correspond to ideals of $\mathfrak{g}$ and the center of G corresponds to the center of $\mathfrak{g}$. These relationships can be explained via the Ad and ad maps which we now introduce.

Let $I_g : G \to G$ by $I_g(h) = ghg^{-1}$. The differential of I_g, which we denote by $Ad(g)$, is an automorphism of $\mathfrak{g}$ such that the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad(g)}"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "I_g"'] & G \end{tikzcd}$$

commutes.

We can repeat this process for any element of G. This leads us to define a map Ad : $G \to$ Aut$(\mathfrak{g})$. We call this map the **adjoint representation of G** on $\mathfrak{g}$. But both G and Aut$(\mathfrak{g})$ are Lie groups, so we have in turn the differential of Ad, denoted ad, a map from $\mathfrak{g}$ to End$(\mathfrak{g})$. It can be shown that for $X, Y \in \mathfrak{g}$, ad$(X)Y = [X, Y]$

and that the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathrm{End}(\mathfrak{g}) \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}"'] & \mathrm{Aut}(\mathfrak{g}) \end{tikzcd}$$

commutes. We call ad the **adjoint representation of $\mathfrak{g}$.**

Since Ad and ad originated from the conjugation map I_g they capture information about commutativity in G and $\mathfrak{g}$. Thus we see that normal subgroups in G correspond to ideals in $\mathfrak{g}$ follows. Suppose that H is a normal subgroup of G so $I_g(h) \in H$ for all $g \in G$. This holds if and only if $\mathrm{Ad}(g)X \in \mathfrak{h}$ for all $g \in G, X \in \mathfrak{h}$. But from the commutativity of the last diagram, we know this is true if and only if $\mathrm{ad}Y(X) \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, that is, if $[Y, X] \in \mathfrak{h}$. But this is precisely the condition that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

Using similar proofs it can be shown that G is abelian if and only if $\mathfrak{g}$ is abelian and the center of G corresponds to the center of $\mathfrak{g}$. Furthermore, if H is a subgroup of G with associated Lie subalgebra $\mathfrak{h}$, then the centralizer of H in G corresponds to the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Finally, if $\mathfrak{h}$ and $\mathfrak{k}$ are two subalgebras of $\mathfrak{g}$ such that $\mathfrak{h} \oplus \mathfrak{k}$ is a Lie algebra direct sum (that is, $[\mathfrak{h}, \mathfrak{k}] = 0$), then the associated Lie subgroup $H \times K$ is a direct product of groups.

It is worth noting that while each Lie group has a unique associated Lie algebra, the converse is not true. It is possible for two different Lie groups to have isomorphic Lie algebras. For example, both $SO(n)$ and its universal cover $\mathrm{Spin}(n)$ have the same Lie algebra $\mathfrak{so}(\mathrm{n})$. However, suppose that G and G' are two Lie groups with corresponding Lie algebras. The Fundamental Theorem of Lie states that $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic if and only if G and G' are **locally isomorphic**. This means that there exist neighborhoods $\mathcal{U}$ of the identity $e \in G$ and $\mathcal{U}'$ of the identity $e' \in G'$ and a diffeomorphism $\phi : \mathcal{U} \to \mathcal{U}'$ such that whenever $x, y, xy \in \mathcal{U}$, $\phi(xy) = \phi(x)\phi(y)$. In other words, though a particular Lie algebra may have more than one associated Lie group, these Lie groups must have the same local structure.

2.4 More Definitions

In the final section of this chapter, we define a few more terms which will be relevant to later chapters.

For a Lie algebra $\mathfrak{g}$, we define the **commutator series** of $\mathfrak{g}$ by

$$\mathfrak{g}^0 = \mathfrak{g} \supseteq \mathfrak{g}^1 = [\mathfrak{g}^0, \mathfrak{g}^0] \supseteq \cdots \supseteq \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}^{k-1}] \supseteq \dots \tag{2.5}$$

We say that $\mathfrak{g}$ is **solvable** if $\mathfrak{g}^k = 0$ for some k.

We may also define the **lower central series** for $\mathfrak{g}$ as

$$\mathfrak{g}_0 = \mathfrak{g} \supseteq \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}_0] \supseteq \cdots \supseteq \mathfrak{g}_k = [\mathfrak{g}, \mathfrak{g}_k] \supseteq \dots \tag{2.6}$$

and say that $\mathfrak{g}$ is **nilpotent** if $\mathfrak{g}_k = 0$ for some k.

Notice that if $\mathfrak{g}$ is solvable, then the last nonzero $\mathfrak{g}^j$ in the commutator series is an abelian ideal. Similarly, if $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ has nonzero center. In contrast, we say that $\mathfrak{g}$ is **simple** if $\mathfrak{g}$ is nonabelian and has no proper nonzero ideals. We say $\mathfrak{g}$ is **semisimple** if it has no nonzero solvable ideals. In particular, if $\mathfrak{g}$ is semisimple, it has 0 center and may be written as a Lie algebra direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ where each $\mathfrak{g}_j$ is a simple ideal in $\mathfrak{g}$.

Example 2.4.1. The Lie algebras $\mathfrak{so}$(n), $\mathfrak{su}$(n), and $\mathfrak{sp}$(n) are all examples of real simple Lie algebras.

We say that a Lie group is **solvable**, **nilpotent**, **semisimple**, or **simple** if its associated Lie algebra is solvable, nilpotent, semisimple, or simple respectively.

Example 2.4.2. The Lie groups $SO(n)$, $\mathrm{Spin}(n)$, $SU(n)$, and $Sp(n)$ associated to the Lie algebras in Example 2.4.1 are simple Lie groups. They are known as the classical compact simple Lie groups.

Finally, we say that a Lie algebra $\mathfrak{g}$ is **reductive** if for each ideal $\mathfrak{a} \subset \mathfrak{g}$ there exists a corresponding ideal $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{g}$ equals the Lie algebra direct sum $\mathfrak{a} \oplus \mathfrak{b}$. It is not hard to show that any reductive Lie algebra may be written $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}_{\mathfrak{g}}$ where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple and $\mathfrak{z}_{\mathfrak{g}}$ is the center of $\mathfrak{g}$. If G is a compact Lie group with Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ is reductive. This implies that every compact Lie group may be written $(G_1 \times G_2 \times \cdots \times G_n \times T)/Z$ where each G_i is simple, T is a torus, and Z is a discrete central subgroup of $G_1 \times G_2 \times \cdots \times G_n \times T$.

Chapter 3

Spectral Geometry Background

Spectral geometry is the branch of geometry which examines the relationship between the spectrum of a Riemannian manifold and the geometry and topology of the manifold. We begin this chapter by giving the basic definitions and facts of spectral geometry. We then survey some examples of isospectral manifolds, focusing on examples which have different local geometry. In the third section of the chapter, we construct the metrics which we will consider in this paper. These metrics arise from linear maps so in the final section of the chapter, we consider how relationships among the linear maps encode relationships among the associated metrics.

3.1 Background and Definitions

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Given a smooth function f on M, we define the **gradient** of f to be the smooth vector field $\mathrm{grad}(f)$ such that

$$g(\mathrm{grad}(f), v) = v(f) \tag{3.1}$$

for all smooth vector fields v.

On the other hand, given a vector field v on M, we define the real-valued function, $\mathrm{div}(v)$, known as the **divergence** of v, by

$$\mathrm{div}(v)(p) = \mathrm{tr}(w \mapsto \nabla_w v), \tag{3.2}$$

where w ranges over T_pM.

With these definitions in hand, we are ready for the main definition of spectral geometry.

For any C^k, $k \geq 2$, function f on M we define the **Laplacian** of f, denoted Δf, by

$$\Delta f = -\mathrm{div}(\mathrm{grad} f) \tag{3.3}$$

This leads us to consider the following eigenvalue problems.

Closed eigenvalue problem: Suppose M is a compact, connected manifold without boundary. Find all real numbers λ such that there exists some nonzero function $f \in C^2(M)$ satisfying

$$\Delta f = \lambda f. \tag{3.4}$$

Neumann eigenvalue problem: Suppose M is a compact, connected manifold with boundary. Let ν be the outward pointing normal vector field on ∂M. Find all real numbers λ such that there exists some nonzero function $f \in C^2$ satisfying the eigenvalue Equation 3.4 and

$$\nu f = 0. \tag{3.5}$$

Dirichlet eigenvalue problem: Suppose M is a compact, connected manifold with boundary. Find all real numbers λ such that there exists some nonzero function

$f \in C^2$ satisfying Equation 3.4 and

$$f_{|\partial M} = 0 \tag{3.6}$$

In each of these cases, a solution λ is known as an **eigenvalue** of the Laplacian, and a nontrivial function f satisfying $\Delta f = \lambda f$ is called an **eigenfunction** of λ. The set of all eigenfunctions associated to a given eigenvalue is known as the **eigenspace** of λ.

Theorem 3.1.1. *For each of the eigenvalue problems listed above, the set of eigenvalues forms a discrete sequence*

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots \uparrow \infty, \tag{3.7}$$

where the eigenspace of each eigenvalue is finite dimensional. Eigenspaces of distinct eigenvalues are orthogonal in $L^2(M)$, and there exists a basis of eigenfunctions for $L^2(M)$. Finally, each eigenfunction is smooth on $\overline{M}$.

The sequence given by Theorem 3.1.1 is known as the **(Laplace) spectrum** of (M, g). We say that two manifolds (M, g) and (M', g') are **isospectral** if they share the same spectrum.

We will ultimately be interested in continuous families of isospectral manifolds. In particular, suppose M is a smooth manifold with metric g_0. A **nontrivial isospectral deformation** of g_0 is a continuous family $\mathcal{F}$ of metrics on M such that for any metric $g \in \mathcal{F}$, (M, g_0) and (M, g) are isospectral but not isometric. We say that a deformation is **multidimensional** if the parameter space of $\mathcal{F}$ is of dimension greater than 1.

3.2 A Survey of Isospectral Manifolds with Different Local Geometry

We now survey some examples of isospectral manifolds, with a focus on examples having different local geometry. This list is not complete but is meant to give a sense of history as well as a to point out some specific local geometric invariants which are not spectrally determined. For manifolds with boundary, the word isospectral indicates both Neumann and Dirichlet isospectral.

The industry of producing examples of isospectral manifolds began in 1964 when Milnor exhibited a pair of 16-dimensional flat tori which are isospectral but not isometric. Several years later, in the early 1980's, new examples began to appear sporadically. These included pairs of Riemann surfaces and pairs of 3-dimensional hyperbolic manifolds [Vig80], spherical space forms [Ike80], and the first examples of continuous isospectral deformations [GW84].

In 1985, Sunada gave the first unified approach for constructing isospectral examples [Sun85]. With the use of representation theoretic techniques, the method described a program for taking quotients of a given manifold so that the resulting manifolds were isospectral. Sunada's original theorem and subsequent generalizations ([Ber92], [Ber93], [DG89], [Pes96], [Sut02]) explained most of the previously known isospectral examples and led to a wide variety of new examples. See, for example, [BGG98], [Bus86], and [GWW92]. Until a recent generalization by Sutton [Sut02], each example produced using the Sunada method had a common Riemannian cover, which implied that while the manifolds exhibited different global geometry, each pair or family had the same local geometry.

In 1991, using direct computation, Szabó discovered the first examples of isospectral manifolds with different local geometry (these examples were published later in

[Sza99]). The manifolds had boundaries and were diffeomorphic to the product of an eight-dimensional ball and a three-dimensional torus. Inspired in part by Szabó's examples, Gordon produced the first examples of closed isospectral manifolds with different local geometry [Gor93] and then, in a series of papers, generalized the construction to the following principal based on torus actions. Recall that if $\pi : M \to N$ is a submersion, for $p \in M$, the subspace $\ker(\pi_{*p})$ of T_pM is called the vertical space at p and its orthogonal complement is called the horizontal space at p. We say that π is a **Riemannian submersion** if π_* maps the horizontal space at p isometrically onto $T_{\pi(p)}N$. We say that the fibers of a submersion are **totally geodesic** if geodesics in M which start tangent to a fiber remain in the fiber.

Theorem 3.2.1. (Gordon) *Let T be a torus and suppose (M, g) and (M', g') are two principal T-bundles such that the fibers are totally geodesic flat tori. Suppose that for any subtorus $K \subset T$ of codimension 0 or 1, the quotient manifolds $(M/K, \overline{g})$ and $(M'/K, \overline{g}')$, where $\overline{g}$ and $\overline{g}'$ are the induced submersion metrics, are isospectral. Then (M, g) and (M', g') are isospectral.*

This theorem is the basis for what has become known as the "submersion technique". Gordon's initial application of Theorem 3.2.1 was to give a sufficient condition for two compact nilmanifolds (discrete quotients of nilpotent Lie groups) to be isospectral.

In 1997, Gordon and Wilson furthered the development of the submersion technique when they constructed the first examples of *continuous* families of isospectral manifolds with different local geometry [GW97]. The base manifolds were products of n-dimensional balls with r-dimensional tori ($n \geq 5$, $r \geq 2$), realized as domains within nilmanifolds. Here Gordon and Wilson proved a general principle for local nonisometry. They were also able to exhibit specific examples of isospectral deformations of manifolds with boundary for which the eigenvalues of the Ricci tensor (which, in this setting, were constant functions on each manifold) deformed nontrivially. It was later proven in [GGS+98] that the boundaries $S^n \times T^r$ of the manifolds in [GW97] were also examples of isospectral manifolds. These were closed manifolds which were not locally homogeneous. A general abstract principle was given for nonisometry but specific examples were also produced for which the maximum scalar curvature changed throughout the deformation, thereby proving maximal scalar curvature is not a spectral invariant.

Expanding on the ideas of [GGS+98], Schueth produced the first examples of simply connected closed isospectral manifolds. In fact, Schueth even produced continuous families of such manifolds [Sch99]. Schueth's basic principal was to embed the torus T^2 into a larger, simply connected Lie group (e.g. $SU(2) \times SU(2) \simeq S^3 \times S^3$) and expand the metric in order to find isospectral metrics on $S^4 \times SU(2) \times SU(2) \simeq S^4 \times S^3 \times S^3$. Since the torus was embedded in the group, the torus action on the manifold was the natural group action. Schueth's examples were not locally homogeneous. For these examples, the critical value of the scalar curvature changed throughout the deformation, proving the manifolds were not locally isometric. Furthermore, by examining heat invariants related to the Laplacian on one-forms, Schueth was able to prove that these examples were isospectral on functions but not on one-forms.

It is worth noting that at the time of Schueth's examples of simply connected, closed, locally nonisometric manifolds, it was believed that the Sunada method was incapable of producing such manifolds. This was due to the fact that until then, all generalizations of the Sunada's theorem involved quotients by discrete subgroups of isometries. However, Sutton recently proved a generalization which allowed quotients by connected subgroups, and used this generalization to produce isospectral pairs of simply connected, closed manifolds with different local geometry [Sut02].

Schueth continued to capitalize on the notion of embedding the torus in a larger group in her habilitation thesis [Sch01a]. In this case, Schueth specialized Gordon's Theorem 3.2.1 to compact Lie groups in order to produce one-dimensional isospectral deformations of each of $SO(n) \times T^2$ $(n \geq 5)$, $\mathrm{Spin}(n) \times T^2$, $(n \geq 5)$, $SU(n) \times T^2$ $(n \geq 3)$, $SO(n)$ $(n \geq 8)$, $\mathrm{Spin}(n)$ $(n \geq 8)$, and $SU(n)$ $(n \geq 6)$. Here the metrics were left-invariant so the manifolds were homogeneous. As with many previous examples, Schueth's metrics were constructed from linear maps j into the Lie algebra of the Lie group in question. (We will see a detailed exposition of this construction in Section 3.3). In order to prove nonisometry, Schueth expressed the norm of the Ricci tensor in terms of the associated linear j-map. Each of Schueth's examples arose as the flow of a vector field on the space of linear j-maps chosen so that the resulting metrics were isospectral but so that the norm of the Ricci tensor varied through the deformation. These were the first examples of irreducible simply connected isospectral manifolds and the metrics in these examples could be taken arbitrarily close to the bi-invariant metric. (In contrast, Schueth also proved that the bi-invariant metric itself is spectrally rigid within in the class of left-invariant metrics. In other words, any isospectral deformation of left-invariant metrics which contains the bi-invariant metric must be trivial.)

These particular examples of Schueth's were the inspiration for this paper. We will use Schueth's specialization of Theorem 3.2.1 to produce our metrics. However, in this paper we will produce *multidimensional* families of metrics and will develop a general nonisometry principal for families of metrics arising from linear j-maps. Furthermore, we will expand the class of Lie groups for which such exist to include all of the classical compact simple Lie groups of large enough dimension.

In [Sch01a], Schueth also produced examples of isospectral metrics on $S^2 \times T^2$, thereby constructing the lowest dimensional examples to date of isospectral manifolds having different local geometry. These metrics were based on similar, higher dimensional examples constructed in [GGS+98], but were achieved by dropping an unnecessary requirement on the metric construction.

By weakening the hypothesis in Theorem 3.2.1 that the fibers of the torus action be totally geodesic, Gordon and Szabó were able to exhibit a variety of new types of isospectral manifolds [GS02b]. Among the examples were a pair of manifolds with boundary, one of which was Einstein (constant Ricci curvature) and the other not, a pair of manifolds with boundary, one of which had parallel curvature tensor and the other not, and continuous families of isospectral negatively curved manifolds with boundary. The last example was in contrast with a result of Guillemin and Kahzdan (for two dimensions) and Croke and Sharafutdinov (for higher dimensions) which state that nontrivial isospectral deformations of closed negatively curved manifolds are impossible [GK80], [CS98].

Inspired by the examples in [Sch01a], Gordon produced examples of continuous families of isospectral metrics on spheres of dimension 8 and higher and balls of dimension 9 and higher [Gor01]. In the construction of these examples, Gordon dropped the requirement that the torus action be free. The metrics on S^n were T^2-invariant metrics such that an open submanifold of S^n was foliated by principal orbits isometric to manifolds studied in [GW97] and [GGS+98]. Gordon's metrics on the sphere (resp. ball) can be chosen arbitrarily close to the round (resp. flat) metric.

Slightly preceding [Gor01], Szabó also devised a method for constructing pairs of isospectral metrics on spheres [Sza01]. The two methods and examples were distinct as Szabó's method relied on an analysis of function spaces. One very interesting example arising from Szabó's work was a pair of isospectral manifolds, one of which was homogeneous and the other not.

Schueth immediately followed up on [Gor01] by reducing the dimensions of the spheres and balls. [Sch01b] contains examples of pairs of isospectral metrics on S^5 and B^6 as well as continuous isospectral deformations of S^7 and B^8. In each of these examples, the isospectral metrics can be chosen to be the round metric outside an open subset of arbitrarily small volume. In this case, Schueth proved nonisometry by deriving a general sufficient nonisometry condition.

Most recently, Gordon and Schueth have constructed conformally equivalent metrics $\phi_1 g$ and $\phi_2 g$ on spheres S^n and balls B^{n+1} ($n \geq 7$) and on certain compact simple Lie groups [GS02a]. They also showed that the conformal factors ϕ_1 and ϕ_2 were isospectral potentials for the Schrödinger operator $h^2\Delta + \phi$ for all h. There were previously known examples of isospectral conformally equivalent metrics (e.g. [BG90], [BPY89], [Sch01a]) and of isospectral potentials (e.g. [Bro87]), but the examples in [GS02a] represent the first examples on simply connected closed manifolds.

3.3 A Metric Construction

Here we construct the metrics that we will consider in this paper. In what follows, let $\mathfrak{h}$ denote the Lie algebra of the torus T^2, suppose G_n is one of $SO(n)$, Spin(n), $SU(n)$, or $Sp(n)$, and let $\mathfrak{g}_n$ denote the associated Lie algebra $\mathfrak{so}(\mathrm{n})$, $\mathfrak{su}(\mathrm{n})$, or $\mathfrak{sp}(\mathrm{n})$. The metrics which we construct are based on linear maps from $\mathfrak{h}$ into $\mathfrak{g}_n$. The key ideas of the construction are to embed $G_n \times T^2$ into a larger group and then use j to redefine orthogonality on $\mathfrak{g}_n \oplus \mathfrak{h}$. This construction is due to Schueth in [Sch01a].

First we embed $G_n \times T^2$ into a larger group $\mathrm{G_{n+p}}$, where p depends on G_n as we shall see below. All of the work will be done at the Lie algebra level and we will use the fundamental correspondence between Lie algebras and Lie groups to achieve the desired embedding.

- $\mathfrak{g}_n = \mathfrak{so}(\mathrm{n})$: Consider $\mathfrak{h}$ a subalgebra of $\mathfrak{so}(n+4)$ (i.e. $p = 4$) with elements of the form

$$\begin{bmatrix} 0_n & & & & \\ & 0 & \alpha & & \\ & -\alpha & 0 & & \\ & & & 0 & \beta \\ & & & -\beta & 0 \end{bmatrix}, \tag{3.8}$$

where $\alpha, \beta \in \mathbb{R}$. We also consider $\mathfrak{so}(\mathrm{n})$ a subalgebra of $\mathfrak{so}(n+4)$ as the set of elements

$$\begin{bmatrix} A & \\ & 0_4 \end{bmatrix}, \tag{3.9}$$

where $A \in \mathfrak{so}(\mathrm{n})$. In this case, $\mathfrak{so}(\mathrm{n}) \oplus \mathfrak{h}$ is a Lie algebra direct sum so we may consider the direct product $SO(n) \times T^2$ (resp. $\mathrm{Spin}(n) \times T^2$) a Lie subgroup of $SO(n+4)$ (resp. $\mathrm{Spin}(n+4)$).

- $\mathfrak{g}_n = \mathfrak{su}(\mathrm{n})$: We have $\mathfrak{h}$ a subalgebra of $\mathfrak{su}(n+3)$ (i.e. $p = 3$) consisting of elements

$$\begin{bmatrix} 0_n & & & \\ & \alpha i & & \\ & & \beta i & \\ & & & -(\alpha+\beta)i \end{bmatrix}, \tag{3.10}$$

with $\alpha, \beta \in \mathbb{R}$ and $\mathfrak{su}(\mathrm{n})$ the subalgebra of $\mathfrak{su}(n+3)$ consisting of elements

$$\begin{bmatrix} A & \\ & 0_3 \end{bmatrix}, \tag{3.11}$$

$A \in \mathfrak{su}(\mathrm{n})$. We then have the Lie algebra direct sum $\mathfrak{su}(\mathrm{n}) \oplus \mathfrak{h}$ corresponds to a Lie subgroup of $SU(n+3)$ which is the direct product $SU(n) \times T^2$

- $\mathfrak{g}_n = \mathfrak{sp}(\mathrm{n})$: Consider $\mathfrak{h}$ the subalgebra of elements of $\mathfrak{sp}(n+2)$ (i.e. $p=2$) of the form

$$\begin{bmatrix} 0_n &&&&& \\ & \alpha i &&&& \\ && \beta i &&& \\ &&& 0_n && \\ &&&& -\alpha i & \\ &&&&& -\beta i \end{bmatrix}, \tag{3.12}$$

  with $\alpha, \beta \in \mathbb{R}$. We have $\mathfrak{sp}(\mathrm{n})$ a subalgebra of $\mathfrak{sp}(n+2)$ as the set of elements

$$\begin{bmatrix} A && B & \\ & 0_2 && 0_2 \\ -\overline{B} && \overline{A} & \\ & 0_2 && 0_2 \end{bmatrix} \tag{3.13}$$

  where A, B are $n \times n$ complex matrices, A is skew Hermitian, and B is symmetric. Thus the direct product $Sp(n) \times T^2$ is contained in $Sp(n+2)$ as a Lie subgroup.

Next we define a family of inner products on $\mathfrak{g}_{n+p}$. The inner products will all be based on linear maps, so we have the following definition.

Definition 3.3.1. A *j-map* is a linear map $j : \mathfrak{h} \to \mathfrak{g}_n$. We denote the space of all j-maps into $\mathfrak{g}_n$ by L.

Let g_0 be the inner product on $\mathfrak{g}_{n+p}$ arising from the bi-invariant metric on G_{n+p}. Given a linear map $j : \mathfrak{h} \to \mathfrak{g}_n \subset \mathfrak{g}_{n+p}$ we have $j^t : \mathfrak{g}_{n+p} \to \mathfrak{h}$ defined by

$$g_0(j^t(X), Z) = g_0(X, j(Z)) \tag{3.14}$$

for all $X \in \mathfrak{g}_{n+p}$, $Z \in \mathfrak{h}$. In other words, j^t is the adjoint of j with respect to g_0 on $\mathfrak{g}_{n+p}$ and g_0 restricted to $\mathfrak{h}$. Let g_j be the inner product on $\mathfrak{g}_{n+p}$ given by $g_j = (Id + j^t)^* g_0$.

Finally, we complete the metric construction by taking the left-invariant metric on G_{n+p} associated to g_j. We will denote this metric by g_j as well.

For fixed j, we catalogue some useful attributes of the inner product g_j. A key observation is that if $X \in \mathfrak{g}_{\mathrm{n+p}}$ is g_0-orthogonal to the image of j, then $j^t(X) = 0$. This is easy to see since for each $Z \in \mathfrak{h}$,

$$g_0(j^t(X), Z) = g_0(X, j(Z)) = 0. \tag{3.15}$$

Since the image of j is contained $\mathfrak{g}_{\mathrm{n}}$, we have that j restricted to the $\mathfrak{g}_{\mathrm{n}}^{\perp_{\mathrm{g}_0}}$ equals 0.

It is useful to decompose $\mathfrak{g}_{\mathrm{n+p}}$ into orthogonal subspaces $\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h}$ and $(\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}}$ and consider g_j on each. Notice that $(\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g_j}}} = (\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}}$ as follows. Let $X \in \mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h}$ and let $Y \in (\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}}$. Then since $j^t(X) \in \mathfrak{h}$ and $j^t(Y) = 0$,

$$g_j(X, Y) = g_0(X + j^t(X), Y + j^t(Y)) \tag{3.16}$$

$$= g_0(X + j^t(X), Y) \tag{3.17}$$

$$= 0. \tag{3.18}$$

Furthermore, g_j restricted to $(\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}}$ is equal to g_0. Indeed, if $X, Y \in (\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}}$,

$$g_j(X, Y) = g_0(X + j^t(X), Y + j^t(Y)) \tag{3.19}$$

$$= g_0(X, Y) \tag{3.20}$$

On the other hand, compare g_0 with g_j on $\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h}$. We already know that j^t restricted to $\mathfrak{g}_{\mathrm{n}}^{\perp_{\mathrm{g}_0}}$ equals 0. In particular, j^t equals 0 on $\mathfrak{h}$ so an argument similar to the one above shows that g_j restricted to $\mathfrak{h}$ equals g_0. We calculate $\mathfrak{h}^{\perp_{g_j}}$ in $\mathfrak{g}_n \oplus \mathfrak{h}$. Suppose $X \in \mathfrak{g}_n$ and consider $X - j^t(X) \in \mathfrak{g}_n \oplus \mathfrak{h}$. For any $Z \in \mathfrak{h}$,

$$g_j(X - j^t(X), Z) = g_0(X - j^t(X) + j^t(X), Z) \tag{3.21}$$

$$= g_0(X, Y) \tag{3.22}$$

$$= 0. \tag{3.23}$$

Thus we see that the space $S = \{X - j^t(X) | X \in \mathfrak{g}_n\}$ is g_j-orthogonal to $\mathfrak{h}$. Furthermore, for any elements $X - j^t(X), Y - j^t(Y) \in S$,

$$g_j(X - j^t(X), Y - j^t(Y)) = g_0(X - j^t(X) + j^t(X), Y - j^t(Y) + j^t(Y)) \tag{3.24}$$

$$= g_0(X, Y). \tag{3.25}$$

In summary, we have found that g_j differs from g_0 only on $\mathfrak{g}_n \oplus \mathfrak{h}$. On $\mathfrak{g}_n \oplus \mathfrak{h}$ we have used the linear map j to redefine orthogonality. In particular, j determines a subspace $S = \{X - j^t(X) | X \in \mathfrak{g}\}$ which is g_j-orthogonal to $\mathfrak{h}$ and such that g_j restricted to S is linearly isometric to g_0 restricted to $\mathfrak{g}_n$ via the map $X - j^t(X) \mapsto X$.

3.4 Isospectrality and Equivalence

In the final section of this chapter, we define two important algebraic relationships between j-maps: isospectrality and equivalence. These relationships will translate into information about the associated metrics. In particular, we state a theorem by Schueth which tells us that isospectral j-maps give rise to isospectral metrics. Later, in Chapter 5, we will prove that for a general j-map j_0, a family of pairwise nonequivalent j-maps containing j_0 gives rise to a family of metrics which are not isometric to

g_{j_0}. Thus we are able to rephrase the geometric question of finding nontrivial isospectral deformations in terms of the algebraic question of finding continuous families of j-maps which are isospectral but not equivalent.

Definition 3.4.1. Two j-maps, j and j', into $\mathfrak{g}_n$ are called *isospectral* if for each $z \in \mathfrak{h}$ there exists $A_z \in G_n$ such that

$$\mathrm{Ad}(A_z)j(z) = j'(z).$$

We denote this by $j \sim j'$.

Definition 3.4.2. Two j-maps, j and j', into $\mathfrak{g}_n$ are called *equivalent* if there exist $C \in O(\mathfrak{h})$ and $A \in G_n$ such that

$$\mathrm{Ad}(A)j(z) = j'(C(z)) \tag{3.26}$$

for all $z \in \mathfrak{h}$. We denote this by $j \simeq j'$.

Remark 3.4.3. Note that in the case $G_n = SO(n)$, $SU(n)$, or $Sp(n)$, the map $\mathrm{Ad}(A) : \mathfrak{g}_n \to \mathfrak{g}_n$ is given by matrix conjugation. Thus we may rewrite the isospectrality condition as

$$A_z j(z) A_z^{-1} = j'(z) \tag{3.27}$$

and the equivalence condition as

$$Aj(z)A^{-1} = j'(C(z)). \tag{3.28}$$

The following theorem by Schueth is a specialization of Gordon's submersion theorem (Theorem 3.2.1).

Theorem 3.4.4. *Let G be a compact Lie group with Lie algebra $\mathfrak{g} = T_eG$, and let g_0 be a bi-invariant metric on G. Let $H \subset G$ be a torus in G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Denote by $\mathfrak{u}$ the g_0-orthogonal complement of the centralizer $\mathfrak{z}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$. Let λ, $\lambda' : \mathfrak{g} \to \mathfrak{h}$ be two linear maps with $\lambda_{|_{\mathfrak{h} \oplus \mathfrak{u}}} = \lambda'_{|_{\mathfrak{h} \oplus \mathfrak{u}}} = 0$ which satisfy: For every $z \in \mathfrak{h}$ there exists $A_z \in G$ such that A_z commutes with H and $\lambda' = \mathrm{Ad}(A_z)^* \lambda_z$, where $\lambda_z := g_0(\lambda(\cdot), z)$ and $\lambda'_z := g_0(\lambda'(\cdot), z)$. Denote by g_λ and $g_{\lambda'}$ the left-invariant metrics on G which correspond to the scalar products $(\mathrm{Id} + \lambda)^* g_0$ and $(\mathrm{Id} + \lambda')^* g_0$ on $\mathfrak{g}$. Then (G, g_λ) and $(G, g_{\lambda'})$ are isospectral.*

In particular, this theorem tells us that if j and j' are isospectral maps $j, j' : \mathfrak{h} \to \mathfrak{g}_n$, then letting $\lambda = j^t$ and $\lambda' = {j'}^t$, we may conclude that the metrics g_j and $g_{j'}$ on $\mathrm{G_{n+p}}$ defined in Section 3.3 are isospectral.

Chapter 4

Multidimensional Families of j-Maps

In this chapter we will establish the existence of multiparameter families of j-maps into $\mathfrak{so}$(n) ($n = 5, n \geq 7$), $\mathfrak{su}$(n) ($n \geq 4$), and $\mathfrak{sp}$(n) ($n \geq 3$) which are isospectral but not equivalent. Based on the comments in Section 3.4, this result will ultimately lead us to nontrivial isospectral deformations of metrics on $SO(n)$ ($n = 9, n \geq 11$), Spin(n) ($n = 9, n \geq 11$), $SU(n)$ ($n \geq 7$), and $Sp(n)$, ($n \geq 5$).

This chapter is devoted entirely to the statement and proof of Theorem 4.1.1. The chapter is divided into sections according to the steps of the proof.

4.1 Introduction

Throughout this chapter we will let $\mathfrak{h}$ denote the Lie algebra of the two-dimensional torus T^2. Let $[\frac{n}{2}]$ be the largest integer less than or equal to $\frac{n}{2}$.

Theorem 4.1.1. *Suppose $\mathfrak{g}_n$ is one of* $\mathfrak{so}$(n) ($n = 5, n \geq 7$) , $\mathfrak{su}$(n) ($n \geq 4$), *or* $\mathfrak{sp}$(n) ($n \geq 3$). *Let L be the space of all linear maps $j : \mathfrak{h} \to \mathfrak{g}_n$. There exists a*

Zariski open set $\mathcal{O} \subset L$ such that each $j_0 \in \mathcal{O}$ is contained in a d-parameter family of linear maps which are isospectral but not equivalent. Here d depends on $\mathfrak{g}_n$ as follows:

$\mathfrak{g}_n$	d
$\mathfrak{so}(\mathrm{n})$	$d \geq n(n-1)/2 - [\frac{n}{2}]([\frac{n}{2}]+2)$
$\mathfrak{su}(\mathrm{n})$	$d \geq n^2 - 1 - \frac{n^2+3n}{2}$
$\mathfrak{sp}(\mathrm{n})$	$d \geq n^2 - n$

Note that for $\mathfrak{so}(\mathrm{n})$, $d > 1$ when $n = 5$ or $n \geq 7$. For $\mathfrak{su}(\mathrm{n})$, $d = 1$ when $n = 4$ and $d > 1$ when $n \geq 5$. For $\mathfrak{sp}(\mathrm{n})$, $d > 1$ when $n \geq 3$.

Remark 4.1.2. (i.) Theorem 4.1.1 was originally proven for $\mathfrak{so}(\mathrm{n})$ with associated Lie group $O(n)$ in [GW97]. Here we will give the proof for $\mathfrak{su}(\mathrm{n})$ and $\mathfrak{sp}(\mathrm{n})$ with associated Lie groups $SU(n)$ and $Sp(n)$ respectively. The general argument for both Lie algebras will be given and at points where the argument becomes specific by Lie algebra, the proof will be itemized by case.

Proof. For $j \in L$, define the following sets:

$$I_j = \{j' \in L | j \sim j'\} \tag{4.1}$$

and

$$E_j = \{j' \in I_j | j \simeq j'\}. \tag{4.2}$$

Remark 4.1.3. Notice that E_j is the set of all linear maps which are equivalent *and* isospectral to j.

The main work of the proof is to come up with polynomial equations which allow us to define the Zariski open set $\mathcal{O}$. In the process, we will show that for any $j_0 \in \mathcal{O}$, the set $I_{j_0} \cap \mathcal{O}$ is an embedded submanifold of L which can be foliated by sets of the form E_j where $j \in I_{j_0}$. We will then find a d-dimensional submanifold N_{j_0} of I_{j_0}

which contains j_0 and is transverse to the foliation. The submanifold N_{j_0} will be a family of j-maps which are isospectral but not equivalent to j_0.

4.2 An Embedded Submanifold of Isospectral j-Maps

We find a Zariski open set $\mathcal{O}_1 \subset L$ such that for $j_0 \in \mathcal{O}_1$, $I_{j_0} \cap \mathcal{O}_1$ is an embedded submanifold of L.

Let $\mathfrak{g}_n$ be one of $\mathfrak{su}$(n) or $\mathfrak{sp}$(n) and let r denote the size of the matrices in $\mathfrak{g}_n$. (For $\mathfrak{su}$(n), $r = n$ and for $\mathfrak{sp}$(n), $r = 2n$.) Given $k \in \{1, \ldots, r\}$, define a map $T_k : \mathfrak{g}_n \to \mathbb{R}$ by $T_k(C) = \operatorname{tr}(C^k)$. If two elements $C, C' \in \mathfrak{g}_n$ are similar then, since $\operatorname{tr}(ACA^{-1}) = \operatorname{tr}(C)$ and $(ACA^{-1})^m = AC^mA^{-1}$, we have $T_k(C) = T_k(C')$ for all k. In fact, the opposite is true as well. If $C, C' \in \mathfrak{g}_n$ and $T_k(C) = T_k(C')$ for all $k = 1, \ldots, r$, then C and C' must be similar.

Indeed, suppose $T_k(C) = T_k(C')$ for all $k \in \{1, \ldots, r\}$ and recall that all elements of $\mathfrak{g}_n$ are diagonalizable. Since C and C' are diagonalizable, we obtain a system of equations, $s_k = s'_k,\ k = 1, \ldots, r$ where $s_k^{(\prime)} = \sum_{i=1}^r \lambda_i^{(\prime)k}$ and $\lambda_i^{(\prime)}$ is the ith eigenvalue of $C^{(\prime)}$. Letting D denote a diagonal representation of C, we know that for any real number x, $\det(xI - C) = \det(xI - D)$. Thus $\det(xI - C) = (x - \lambda_1)(x - \lambda_2) \ldots (x - \lambda_r)$. If we expand this product we see that $\det(xI - C) = x^r - p_1 x^{(r-1)} + p_2 x^{(r-2)} + \cdots + (-1)^r p_r$, where p_i equals the ith elementary symmetric polynomial $p_i = \sum_{j_1 < j_2 < \cdots < j_i} \lambda_{j_1} \lambda_{j_2} \ldots \lambda_{j_i}$. We have a similar expression for $\det(xI - C')$. According to an argument outlined in [Jac85], it is possible to write each p_i in terms of the functions $s_k,\ k = 1, \ldots, r$. Thus, since $s_k = s'_k$ for all $k = 1, \ldots, r$, we have that $p_i = p'_i$ for all i, and therefore $\det(xI - C) = \det(xI - C')$ for all real numbers x. In particular, C and C' have the same eigenvalues and therefore are similar.

Remark 4.2.1.

- The argument for $\mathfrak{su}$(n) differs here from the argument for $\mathfrak{so}$(n) found in [GW97]. In particular, for $\mathfrak{so}$(n), Gordon and Wilson considered T_k only for even k since T_k is trivial for odd k. This does not hold for $\mathfrak{su}$(n) when $n \geq 3$. Indeed, we know that the trace of any matrix $A \in \mathfrak{su}(n)$ is the sum of its eigenvalues. If we write A in canonical form we know that $\mathrm{tr}(A^k) = \sum a_i^k$ where a_i are the eigenvalues of A. Suppose for example that A has eigenvalues $i, 2i$, and $-3i$. Then clearly, $\mathrm{tr}(A) = 0$ but $\mathrm{tr}(A^3) = -i - 8i + 27i \neq 0$. Thus we must consider the functions T_k for all $k = 1, \ldots, n$.

- The elements of $\mathfrak{sp}$(n) are $2n \times 2n$ matrices whose eigenvalues come in plus and minus complex pairs. Hence the traces of odd powers of elements of $\mathfrak{sp}$(n) are equal to zero and we see that it suffices to consider the functions T_k for all *even* k between 1 and $2n$.

Next, we define a map $T_k : \mathfrak{h} \times L \to \mathbb{R}$ by $T_k(z, j) = T_k(j(z)) = \mathrm{tr}(j(z)^k)$. We then have that j and j' are isospectral if and only if for each $z \in \mathfrak{h}$, $T_k(z, j) = T_k(z, j')$ for all $k = 1, \ldots, r$.

Suppose that $\{e_1, e_2\}$ is an orthonormal basis for $\mathfrak{h}$ and that we write $\mathfrak{h} = \{se_1 + te_2 : s, t \in \mathbb{R}\}$. Then

$$T_k(se_1 + te_2, j) = \mathrm{tr}(j(se_1 + te_2)^k) \tag{4.3}$$

$$= \mathrm{tr}((sj(e_1) + tj(e_2))^k) \tag{4.4}$$

Since any linear map from $\mathfrak{h} \to \mathfrak{g}_n$ is completely determined by $j(e_1)$ and $j(e_2)$, L may be identified with $\mathfrak{g}_n \times \mathfrak{g}_n$. But $\mathfrak{g}_n$ is coordinatized by its matrix entries. Thus, each T_k is a polynomial function on $\mathfrak{h} \times L$ homogeneous of degree k in each variable

$\mathfrak{h}$ and L.

For $p+q=k$, let $f_{s^pt^q}(j)$ denote the trace of the coefficient of s^pt^q in the expansion of $(sj(e_1)+tj(e_2))^k$. We see that each $f_{s^pt^q}$ is a polynomial function on L and that $T_k(se_1+te_2,j)=T_k(se_1+te_2,j')$ for all $se_1+te_2 \in \mathfrak{h}$ if and only if $f_{s^pt^q}(j)=f_{s^pt^q}(j')$ for all $p+q=k$. Indeed, if $f_{s^pt^q}(j)=f_{s^pt^q}(j')$ for all $p+q=k$, then by the definitions of $f_{s^pt^q}$ and T_k, it is clear that $T_k(se_1+te_2,j)=T_k(se_1+te_2,j')$ for all $se_1+te_2 \in \mathfrak{h}$. On the other hand, if $T_k(se_1+te_2,j)=T_k(se_1+te_2,j')$ for all $se_1+te_2 \in \mathfrak{h}$, then

$$\begin{aligned} s^k(f_{s^k}(j)-f_{s^k}(j'))+s^{k-1}t(f_{s^{k-1}t}(j)-f_{s^{k-1}t}(j'))+\dots \\ +st^{k-1}(f_{st^{k-1}}(j)-f_{st^{k-1}}(j'))+t^k(f_{t^k}(j)-f_{t^k}(j'))=0 \quad (4.5) \end{aligned}$$

for all $s,t\in\mathbb{R}$. In other words, $f_{s^pt^q}(j)=f_{s^pt^q}(j')$ for all $p+q=k$.

For any k, there are $k+1$ functions $f_{s^pt^q}$ associated to T_k.

- For $\mathfrak{su}(\mathrm{n})$, we consider T_k for all $k=1,\dots n$. Since $\sum_{k=1}^n k+1=\sum_{k=1}^n k+\sum_{k=1}^n 1=\frac{n(n+1)}{2}+n=\frac{n^2+3n}{2}$ we conclude that j is isospectral to j' if and only if all $\frac{n^2+3n}{2}$ polynomial functions agree on j and j'. This indicates that we can define a function $F: L\to\mathbb{R}^{\frac{n^2+3n}{2}}$ such that $j\sim j'$ if and only if $F(j)=F(j')$.

- For $\mathfrak{sp}(\mathrm{n})$, we consider T_k for all even k between 1 and $2n$. Since $\sum_{j=1}^n 2j+1=2\sum_{j=1}^n j+n=\frac{2n(n+1)}{2}+n=n^2+2n$, we know that j is isospectral to j' if and only if all n^2+2n polynomial functions agree on j and j'. Hence, we can define a function $F: L\to\mathbb{R}^{n^2+n}$ such that $j\sim j'$ if and only if $F(j)=F(j')$.

Let R be the maximum rank of F. Then for $\mathfrak{su}(\mathrm{n})$, $R\le\frac{n^2+3n}{2}$ and for $\mathfrak{sp}(\mathrm{n})$, $R\le n^2+2n$. Also, F has rank R at a point j if and only if there is some $R\times R$ minor of F_{*j} which has nonzero determinant. In other words, F has rank R at j if and only if the sum of the squares of all determinants of $R\times R$ minors is nonzero.

But this sum of squares is a polynomial, denoted ϕ_F, so the set of points at which F has rank R is a Zariski open set in L. (Notice that the set is nonempty by the choice of R.) We'll name this Zariski open set $\mathcal{O}_1$.

We have $F : L \to \mathbb{R}^{\frac{n^2+3n}{2}}$ (resp. $\mathbb{R}^{n^2+2n}$) defined by polynomials, hence C^∞. For $j_0 \in \mathcal{O}_1$, $F^{-1}(F(j_0)) = I_{j_0}$. Furthermore, by the definition of $\mathcal{O}_1$, F restricted to $\mathcal{O}_1$ has constant rank R. Thus the Implicit Function Theorem implies that $I_{j_0} \cap \mathcal{O}_1$ is a closed, embedded submanifold of $\mathcal{O}_1$ of codimension R. Since $\mathcal{O}_1$ is open in L, $I_{j_0} \cap \mathcal{O}_1$ is an embedded submanifold of L.

4.3 Equivalent j-Maps Within an Isospectral Family

In this section, we find a Zariski open subset $\mathcal{O}_2 \subset L$ such that for $j_0 \in \mathcal{O}_2$ and $j \in I_{j_0}$, the set $E_j \subset I_{j_0}$ is the orbit of a certain group action.

As usual, let G_n be either $SU(n)$ or $Sp(n)$ and let $\mathfrak{g}_n$ denote the corresponding Lie algebras $\mathfrak{su}$(n) and $\mathfrak{sp}$(n). Let r denote the size of the matrices in $\mathfrak{g}_n$. Notice that the group $G_n \times O(\mathfrak{h})$ acts on L by

$$((A, C) \cdot j)(z) = \mathrm{Ad}(A) j(C^{-1}(z)). \tag{4.6}$$

From the definition of equivalence, we see that j' is equivalent to j if and only if $j' = (A, C) \cdot j$ for some $(A, C) \in G_n \times O(\mathfrak{h})$. Now suppose that $j' \in E_j$ (again: so j' is equivalent *and* isospectral to j) and that $j' = (A, C) \cdot j$. In this case we will deduce severe restrictions on what tranformations C could possibly be. In particular, we will show that there is a Zariski open subset $\mathcal{O}_2$ of L such that for j in $\mathcal{O}_2$,

- for $\mathfrak{su}$(n), $j' \in E_j$ if and only if $j' = (A, I_{\mathfrak{h}}) \cdot j$ where $A \in SU(n)$ and $I_{\mathfrak{h}}$ denotes the identity element of $O(\mathfrak{h})$. In other words, we show that E_j is the orbit of j under the action of the subgroup $K = SU(n) \times \{I_{\mathfrak{h}}\}$.

- for $\mathfrak{sp}$(n), $j' \in E_j$ if and only if $j' = (A, \pm I_{\mathfrak{h}}) \cdot j$ where $A \in Sp(n)$. In other words, we show that E_j is the orbit of j under the action of the subgroup $K = Sp(n) \times \{\pm I_{\mathfrak{h}}\}$.

We will also show that:

- for $G_n = SU(n)$, the stability subgroup of K at j is $\{(e^{i\alpha} I_n, I_{\mathfrak{h}}) | (e^{i\alpha})^n = 1\}$ and

- for $G_n = Sp(n)$, the stability subgroup of K at j is $\{(\pm I_{2n}, I_{\mathfrak{h}})\}$.

Again, let $j' \in E_j$ and choose $(A, C) \in G_n \times O(\mathfrak{h})$ such that $j' = (A, C) \cdot j$. Recall that $E_j \subset I_j$ so that all elements of E_j are isospectral to j. Then we have that

$$j \sim (A, C) \cdot j \sim (A^{-1}, I_{\mathfrak{h}}) \cdot (A, C) \cdot j \tag{4.7}$$

(Recall the definition of isospectral maps. We have $(A^{-1}, 1_{\mathfrak{h}}) \cdot (A, C) \cdot j \sim (A, C) \cdot j$ since we can trivially choose A_z to be A^{-1} for all $z \in \mathfrak{h}$.) But

$$(A^{-1}, I_{\mathfrak{h}}) \cdot (A, C) \cdot j(z) = (A^{-1}, I_{\mathfrak{h}}) \cdot \mathrm{Ad}(A) j(C^{-1} z) = j(C^{-1} z) = j \circ C^{-1}(z). \tag{4.8}$$

Thus $j \circ C^{-1} \sim j$. From Section 4.2, we have that $T_k(z, j \circ C^{-1}) = T_k(z, j)$ for all $z \in \mathfrak{h}$ and all $k \in \{1, \ldots, r\}$.

Recall that since $\mathfrak{g}_n$ consists of skew-hermitian matrices (i.e. $c^* = -c$ for all c in $\mathfrak{g}_n$) there is a real inner product on $\mathfrak{g}_n$ given by $\langle c, d \rangle = \mathrm{tr}(cd^*) = -\mathrm{tr}(cd)$. Given a fixed j-map, j, we can relate the inner product on $\mathfrak{g}_n$ back to a semi-inner product, denoted $\langle\cdot,\cdot\rangle_j$, on $\mathfrak{h}$ by $\langle z_1, z_2\rangle_j = \langle j(z_1), j(z_2)\rangle$. In this sense the j-map is translating information about $\mathfrak{g}_n$ back to information about $\mathfrak{h}$ in a way that is unique to the j-map.

Notice that for any $z \in \mathfrak{h}$ $\langle z, z\rangle_j = -T_2(z, j)$. Since $(A, C)\cdot j \in E_j$ implies $T_2(z, j\circ C^{-1}) = T_2(z, j)$ for all $z \in \mathfrak{h}$, we have

$$\langle C^{-1}z, C^{-1}z\rangle_j = -T_2(C^{-1}z, j) = -T_2(z, j\circ C^{-1}) = -T_2(z, j) = \langle z, z\rangle_j. \tag{4.9}$$

In other words, if $(A, C)\cdot j \in E_j$, then C is orthogonal with respect to $\langle\cdot,\cdot\rangle_j$.

Now suppose $\{e_1, e_2\}$ is basis for $\mathfrak{h}$ which is orthonormal with respect to the standard inner product, and let

$$J = \begin{bmatrix} |j(e_1)|^2 & \langle j(e_1), j(e_2)\rangle \\ \langle j(e_1), j(e_2)\rangle & |j(e_2)|^2 \end{bmatrix} \tag{4.10}$$

denote the matrix of $\langle\cdot,\cdot\rangle_j$ with respect to this basis. If J is not diagonal, since it is symmetric with real entries, it is diagonalizable by an orthogonal linear transformation. Thus we will assume J is diagonal. To say that C is orthogonal with respect to $\langle\cdot,\cdot\rangle_j$ is to say that $C^tJC = J$. But since $C \in O(\mathfrak{h})$, this means that $CJ = JC$.

If j is such that J has distinct eigenvalues, it is clear that J will commute with C if and only if C itself is diagonal. Since C must be in $O(\mathfrak{h})$, we know that the only eigenvalues it can have are ± 1. Thus we conclude that as long as J is not a multiple of the 2×2 identity matrix, the only possibilities for C are $\pm I_{\mathfrak{h}}$ and $\pm C_0$, where C_0 is the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right]$.

It is clear that J is a multiple of the identity matrix if and only if the polynomial

$$\phi_{T_2}(j) = (|j(e_1)|^2 - |j(e_2)|^2) + (\langle j(e_1), j(e_2)\rangle)^2 \tag{4.11}$$

equals zero.

- For $\mathfrak{su}$(n), we also know that if $(A, C_0) \cdot j \in E_j$, C_0 must satisfy $T_3(z, j \circ C_0^{-1}) = T_3(z, j)$ for all $z \in \mathfrak{h}$. In other words, it must be that $\mathrm{tr}(j(C_0^{-1}(z))^3) = \mathrm{tr}(j(z)^3)$. In particular, consider $z = e_1 + e_2$. Then, defining

$$\phi_{T_3}(j) = T_3(e_1 + e_2, j) - T_3(e_1 + e_2, j \circ C_0^{-1}) \tag{4.12}$$

we see

$$\begin{aligned}
\phi_{T_3}(j) &= T_3(e_1 + e_2, j) - T_3(e_1 + e_2, j \circ C_0^{-1}) &(4.13)\\
&= \mathrm{tr}(j(e_1 + e_2)^3 - j(C_0(e_1 + e_2))^3) &(4.14)\\
&= \mathrm{tr}(j(e_1 + e_2)^3 - j(e_1 - e_2)^3) &(4.15)\\
&= \mathrm{tr}((j(e_1) + j(e_2))^3) - (j(e_1) - j(e_2))^3) &(4.16)\\
&= \mathrm{tr}(6j(e_1)^2 j(e_2) + 2j(e_2)^3), &(4.17)
\end{aligned}$$

which is a polynomial equation in j. If we can find an example of a j-map such that ϕ_{T_3} does not equal zero then we know that ϕ_{T_3} is not identically zero on L. This will imply that the complement of the set of zeros of ϕ_{T_3} is a Zariski open set in L.

For $\mathfrak{su}$(n) and $n = 3$ choose j such that

$$j(e_1) = \begin{bmatrix} -i & & \\ & -2i & \\ & & 3i \end{bmatrix}, \; j(e_2) = \begin{bmatrix} 5i & 2 & \\ -2 & -4i & \\ & & -i \end{bmatrix}. \tag{4.18}$$

For $n > 3$, augment $j(e_1)$ and $j(e_2)$ with the appropriate number of zeroes.

Notice that in this case, $J = \left[\begin{smallmatrix} 14 & 0 \\ 0 & 50 \end{smallmatrix}\right]$. Then we have that $\phi_{T_3}(j) \neq 0$ as desired. This tells us that for our specific choice of C_0 defined above, $\operatorname{tr}(j(C_0^{-1}(e_1 + e_2))^3) \neq \operatorname{tr}(j(e_1 + e_2)^3)$. In other words, $j \circ C_0^{-1}$ is not isospectral to j so $j \circ C_0^{-1}$ cannot be an element of E_j.

- For $\mathfrak{sp}$(n), and $n = 3$, consider the map j given by

$$j(e_1) = \begin{bmatrix} -i & & & & & \\ & -2i & & & & \\ & & 3i & & & \\ & & & i & & \\ & & & & 2i & \\ & & & & & -3i \end{bmatrix}, j(e_2) = \begin{bmatrix} 5i & 2 & & & & \\ -2 & -4i & & & & \\ & & -i & & & \\ & & & -5i & 2 & \\ & & & -2 & 4i & \\ & & & & & i \end{bmatrix}. \tag{4.19}$$

In this case, $J = \left[\begin{smallmatrix} 28 & 0 \\ 0 & 100 \end{smallmatrix}\right]$. Direct calculation shows that the eigenvalues of $j(e_1) + j(e_2)$ do not equal the eigenvalues of $j(e_1) - j(e_2)$. Thus $j \circ C_0^{-1}$ and j are not isospectral so $j \circ C_0^{-1}$ cannot be an element of E_j. For $n > 3$, augment $j(e_1)$ and $j(e_2)$ with the appropriate number of zeroes. By the same argument as above, we conclude that $j \circ C_0^{-1}$ cannot be an element of E_j.

Furthermore, we may conclude that for this choice of j at least one of

$$\phi_{T_4}(j) = T_4(e_1 + e_2, j) - T_4(e_1 + e_2, j \circ C_0^{-1}) \tag{4.20}$$

$$= \operatorname{tr}((j(e_1) + j(e_2))^4) - (j(e_1) - j(e_2))^4) \tag{4.21}$$

or

$$\phi_{T_6}(j) = T_6(e_1 + e_2, j) - T_6(e_1 + e_2, j \circ C_0^{-1}) \tag{4.22}$$

$$= \mathrm{tr}((j(e_1) + j(e_2))^6) - (j(e_1) - j(e_2))^6) \tag{4.23}$$

does not equal zero. Both ϕ_{T_4} and ϕ_{T_6} are polynomials in j. Let ϕ_{T_i} be ϕ_{T_4} if $\phi_{T_4}(j) \neq 0$. Otherwise let $\phi_{T_i} = \phi_{T_6}$. Since ϕ_{T_i} does not vanish on L, the set of elements where ϕ_{T_i} does not vanish is a Zariski open set in L.

Finally, we have that for $A \in G_n$, $(A, -I_{\mathfrak{h}}) \cdot j(z) = -Aj(z)A^{-1}$. Recall that for $\mathfrak{su}$(n), in order for $(A, C) \cdot j$ to be in E_j, we must have $T_k(j, z) = T_k((A, C) \cdot j, z)$ for all $z \in \mathfrak{h}$ and all $k = 1, \ldots, n$. Since $T_k(j, z) = -T_k((A, -I_{\mathfrak{h}}) \cdot j, z)$ for all odd k, we have that $(A, -I_{\mathfrak{h}}) \cdot j \notin E_j$. However, for $\mathfrak{sp}$(n) it suffices to consider T_k for even k between 1 and $2n$, so we have that both $(A, I_{\mathfrak{h}}) \cdot j$ and $(A, -I_{\mathfrak{h}}) \cdot j$ are elements of E_j.

Thus we have that if $j \in L$ is neither a zero of ϕ_{T_2} nor of ϕ_{T_4}, then

- **for $\mathfrak{su}$(n), $(A, C) \cdot j \in E_j$ if and only if $(A, C) \in K = SU(n) \times \{I_{\mathfrak{h}}\}$.**
- **for $\mathfrak{sp}$(n), $(A, C) \cdot j \in E_j$ if and only if $(A, C) \in K = Sp(n) \times \{\pm I_{\mathfrak{h}}\}$.**

Recall that we would now like to show that

- the stability subgroup of the action of $K = SU(n) \times \{I_{\mathfrak{h}}\}$ on L at j is $\{(e^{i\alpha} I_n, I_{\mathfrak{h}}) | (e^{i\alpha})^n = 1\}$ and
- the stability subgroup of the action of $K = Sp(n) \times \{\pm I_{\mathfrak{h}}\}$ on L at j is $\{(\pm I_{2n}, I_{\mathfrak{h}})\}$.

Remark 4.3.1. For $\mathfrak{sp}(\mathrm{n})$, since C must equal $\pm I_{\mathfrak{h}}$, by Remark 3.4.3 we see that $(A, C) \cdot j = j$ if and only if either $C = I_{\mathfrak{h}}$ and A commutes with $j(z)$ for all z, or $C = -I_{\mathfrak{h}}$ and A anticommutes with $j(z)$ for all z.

Let $r = n$ or $2n$ for $\mathfrak{su}(\mathrm{n})$ and $\mathfrak{sp}(\mathrm{n})$ respectively. Define complex linear maps ϕ_j and $\tilde{\phi}_j$ from $M_r(\mathbb{C})$ to $M_r(\mathbb{C}) \times M_r(\mathbb{C})$ by

$$\phi_j(A) := (j(e_1)A - Aj(e_1), j(e_2)A - Aj(e_2)) \tag{4.24}$$

and

$$\tilde{\phi}_j(A) := (j(e_1)A + Aj(e_1), j(e_2)A + Aj(e_2)). \tag{4.25}$$

Saying that ϕ_j has one-dimensional kernel and that $\tilde{\phi}_j$ is injective is the same as saying that the stability subgroup of K is as above. Indeed, if ϕ_j has one-dimensional kernel, then we know that if $A \in \ker\phi_j$, then A must be a constant multiple of the identity. But if we restrict ϕ_j to

- $SU(n)$, A must be of the form $e^{i\alpha} I_n$ where $(e^{i\alpha})^n = 1$, or
- $Sp(n)$, A must be equal to $\pm I_{2n}$.

From Remark 4.3.1, we conclude that the stability subgroup of K is as desired.

Since ϕ_j and $\tilde{\phi}_j$ are both complex linear maps, we can express each as a $2r^2 \times r^2$ matrix. To say that ϕ_j has one-dimensional kernel is to say that the matrix of ϕ_j has rank $r^2 - 1$, a polynomial condition on j. (The sum of the squares of the determinants of all $(r^2 - 1) \times (r^2 - 1)$ minors must be nonzero for the matrix to have rank $r^2 - 1$.) Similarly, to say that $\tilde{\phi}_j$ is injective is to say that the matrix of $\tilde{\phi}_j$ has rank r^2, also a polynomial condition on j. We call these polynomials ϕ and $\tilde{\phi}$ respectively.

We would like to show that both ϕ and $\tilde{\phi}$ determine Zariski open subsets of L.

- We construct an example of a map $j : \mathfrak{h} \to \mathfrak{su}(\mathrm{n})$ ($n \geq 3$) on which neither ϕ nor $\tilde{\phi}$ vanishes. (Note that it actually suffices to check only that ϕ does not vanish on j.) Let $\{e_1, e_2\}$ be an orthonormal basis of $\mathfrak{h}$. Every element of $\mathfrak{su}(\mathrm{n})$ is a normal linear transformation and therefore is diagonalizable with an orthonormal set of eigenvectors. (Here, orthonormal refers to the complex inner product on $\mathbb{C}^3$.) Furthermore, since each element of $\mathfrak{su}(\mathrm{n})$ is traceless, we know that the sum of its eigenvalues must be zero.

  Choose $j(e_1)$ so that it has an orthonormal set of eigenvectors $\{v_1, \ldots, v_n\}$ with respective eigenvalues $\{a_1, \ldots, a_n\}$. Choose $j(e_1)$ so that $a_1, \ldots, a_n$ are distinct, nonzero, and none is the negative of any of the others. Choose $j(e_2)$ similarly with eigenvectors $\{w_1, \ldots, w_n\}$ and eigenvalues $\{b_1, \ldots, b_n\}$ so that $b_1, \ldots, b_n$ are distinct, nonzero, and none is the negative of any of the others. In addition, choose $j(e_2)$ so that each w_i is a nontrivial linear combination of all of the eigenvectors of $j(e_1)$.

  Suppose that there is some element $S \in M_n(\mathbb{C})$ which commutes with $j(e_1)$. In this case, we can easily see that S must preserve the eigenspaces of $j(e_1)$: let v be an eigenvector of $j(e_1)$ with eigenvalue a_i. Then

$$j(e_1)Sv = Sj(e_1)v = Sa_iv = a_iSv. \tag{4.26}$$

  The fact that we chose distinct eigenvalues for $j(e_1)$ implies that at most S can expand or contract in the directions of the eigenvectors of $j(e_1)$ but can't rotate the space at all. Suppose for now that S is not a multiple of the identity. By an argument similar to the one above, if S commutes with $j(e_2)$, it must preserve the eigenspaces of $j(e_2)$. However, we chose the eigenvectors of $j(e_2)$

so that each was a nontrivial linear combination of all n of the eigenvectors of $j(e_1)$. In the case that S is not a multiple of the identity, it won't preserve the eigenspaces of $j(e_2)$, a contradiction. Therefore, S must be a multiple of the identity so ϕ_j has one-dimensional kernel and hence $\phi(j) \neq 0$.

Now suppose that S anticommutes with $j(e_1)$. Then S takes each eigenspace to the negative eigenspace as follows. Suppose v is an eigenvector of $j(e_1)$ with eigenvalue a_i. Then

$$j(e_1)Sv = -Sj(e_1)v = -Sa_iv = -a_iSv. \tag{4.27}$$

By our choice of $j(e_1)$ none of the eigenvalues of $j(e_1)$ is the negative of either of the others. Therefore, Sv must equal zero. Since the eigenvectors of $j(e_1)$ form an orthonormal basis, we see that S must be trivial. In other words $\tilde{\phi}_j$ in injective so $\tilde{\phi}(j) \neq 0$.

- For $\mathfrak{sp}$(n) we construct two examples of j-maps: one such that $\phi(j) \neq 0$ and one such that $\tilde{\phi}(j) \neq 0$. Recall that the eigenvalues of $\mathfrak{sp}$(n) come in plus/minus pairs.

  The example j for which $\phi(j) \neq 0$ is similar to the example for $\mathfrak{su}$(n). Choose $j(e_1)$ with eigenvalues $\{a_1, \ldots, a_n, -a_1, \ldots, -a_n\}$ such that $a_1, \ldots, a_n$ are distinct, nonzero, and none is the negative of any of the others. Choose $j(e_2)$ such that it has nonzero, distinct eigenvalues $\{b_1, \ldots, b_n, -b_1, \ldots, -b_n\}$ where none of $b_1, \ldots, b_n$ is the negative of any of the others. Choose $j(e_2)$ also so that each eigenvector of $j(e_2)$ is a nontrivial linear combination of all of the eigenvectors of $j(e_1)$. An argument identical to that given for $\mathfrak{su}$(n) shows that if $S \in M_{2n}(\mathbb{C})$ commutes with both $j(e_1)$ and $j(e_2)$, then S must be a multiple of the identity.

Thus $\phi(j) \neq 0$.

Now we provide an example of a j-map into $\mathfrak{sp}(\mathrm{n})$ for which $\tilde{\phi}$ does not vanish, i.e. an example for which the only $S \in M_{2n}(\mathbb{C})$ which anticommutes with both $j(e_1)$ and $j(e_2)$ is $S = 0$.

Choose $j(e_1)$ with orthogonal eigenvectors $\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$ and matching eigenvalues $\{a_1, \ldots, a_n, -a_1, \ldots, -a_n\}$ which are nonzero and distinct but for plus/minus pairs. From the argument for $\mathfrak{su}(\mathrm{n})$ we know that if S anticommutes with $j(e_1)$, then S carries each eigenspace of $j(e_1)$ to its negative eigenspace. Thus we have

$$\begin{array}{ccc} S(v_1) = c_{n+1}v_{n+1} & S(v_2) = c_{n+2}v_{n+2} \ldots & S(v_n) = c_{2n}v_{2n} \\ S(v_{n+1}) = c_1 v_1 & S(v_{n+2}) = c_2 v_2 \ldots & S(v_{2n}) = c_n v_n \end{array} \tag{4.28}$$

for some $c_1, \ldots, c_{2n} \in \mathbb{C}$.

Now choose $j(e_2)$ with orthogonal eigenvectors $\{w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n}\}$ and matching eigenvalues $\{b_1, \ldots, b_n, -b_1, \ldots, -b_n\}$ which are nonzero and distinct but for plus/minus pairs. In addition, choose the eigenvectors so that

$$\begin{array}{cccc} w_1 = v_1 & w_2 = v_2 & \ldots & w_n = v_n \\ w_{n+1} = v_{n+2} & \ldots & w_{2n-1} = v_{2n} & w_{2n} = v_{n+1}. \end{array} \tag{4.29}$$

Then by above, for each $t = 1, \ldots n$, $S(w_t) = S(v_t) = c_{n+t}v_{n+t}$ and also $S(w_t) = d_{n+t}w_{n+t} = d_{n+t}v_{n+t+1}$. ($S$ carries each eigenspace to its negative eigenspace.) But by the choice of the $v_i's$, $c_{n+t}v_{n+t}$ is orthogonal to $d_{n+t}v_{n+t+1}$. Thus, for all $t = 1, \ldots, n$, c_{n+t} and d_{n+t} must equal zero. A similar argument shows that for all $t = 1, \ldots, n$, $S(w_{n+t}) = c_{t+1}v_{t+1} = d_t v_t$ and therefore $c_t = d_t = 0$ for all

$t = 1, \ldots, n$. In other words, we have shown that if S anticommutes with both $j(e_1)$ and $j(e_2)$, then S must be 0. Thus $\tilde{\phi}(j) \neq 0$.

To conclude, we know that if j is not a root of ϕ_{T_2} and ϕ_{T_3} (resp. ϕ_{T_i}) then

- **for $\mathfrak{su}$(n), the set E_j is the orbit of j under the action of the subgroup $K = SU(n) \times \{I_{\mathfrak{h}}\}$ and**
- **for $\mathfrak{sp}$(n), the set E_j is the orbit of j under the action of the subgroup $K = Sp(n) \times \{\pm I_{\mathfrak{h}}\}$.**

Furthermore, if j is not a root of either ϕ or $\tilde{\phi}$, then the stability subgroup of j

- **for $SU(n)$ is $\{(e^{i\alpha} I, I_{\mathfrak{h}}) | (e^{i\alpha})^n = 1\}$ and**
- **for $Sp(n)$ is $\{(\pm I_{2n}, I_{\mathfrak{h}})\}$),**

which is finite in both case. Let $\mathcal{O}_2$ denote the Zariski open subset of j-maps which are not roots of any of ϕ_{T_2}, ϕ_{T_3} (resp. ϕ_{T_i}), ϕ or $\tilde{\phi}$.

4.4 d-Parameter Families of Isospectral, Nonequivalent j-Maps

In the final section of this chapter, we combine the results from the previous sections to prove the existence of multidimensional families of j-maps which are isospectral but not equivalent.

As in the previous sections G_n denotes either $SU(n)$ or $Sp(n)$ and $\mathfrak{g}_n$ denotes the corresponding Lie algebra $\mathfrak{su}$(n) or $\mathfrak{sp}$(n) respectively. We let r denote n or $2n$

depending on whether we are considering $\mathfrak{su}$(n) or $\mathfrak{sp}$(n). Let $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$. Let $j_0 \in \mathcal{O}$ and let $P_{j_0} = I_{j_0} \cap \mathcal{O}$. From the work in the first part of the proof, we know that P_{j_0} is an embedded submanifold of L of codimension $R \leq \frac{n^2+3n}{2}$ (resp. $n^2 + 2n$). We now show that when any one of the polynomials defining $\mathcal{O}$ does not vanish at j, it also does not vanish at $k \cdot j$ for any $k \in K = SU(n) \times \{I_{\mathfrak{h}}\}$ (resp. $K = Sp(n) \times \{\pm I_{\mathfrak{h}}\}$). In other words, $\mathcal{O}$ is closed under the action of K. For each polynomial, the argument for $\mathfrak{su}$(n) is the same as the argument for $\mathfrak{sp}$(n).

First consider ϕ_F. Recall from Section 4.2 that F if a vector-valued function on L with component functions $f_{s^p t^q}$. such that $p + q \in \{1, \ldots, r\}$. The functions $f_{s^p t^q}$ are defined so that $f_{s^p t^q}(j) = f_{s^p t^q}(j')$ when j and j' are isospectral. Furthermore, $\phi_F(j) \neq 0$ if F_{*j} has maximum rank among all $j \in L$.

To see that $\phi_F(k \cdot j) \neq 0$ whenever $\phi_F(j) \neq 0$, first consider $k = (A, I_{\mathfrak{h}}) \in K$. Then for any $j \in L$ and $z \in \mathfrak{h}$, $(A, I_{\mathfrak{h}}) \cdot j(z) = \mathrm{Ad}(A)j(z)$ so $(A, I_{\mathfrak{h}}) \cdot j$ is trivially isospectral to j. Hence $f_{s^p t^q}((A, I_{\mathfrak{h}}) \cdot j) = f_{s^p t^q}(j)$ for all $p + q \in \{1, \ldots, r\}$. Thus conjugation by $A \in G$ gives a smooth invertible map from L to L which preserves the component functions of F. This implies that the rank of F at j is equal to the rank of F at $(A, I_{\mathfrak{h}}) \cdot j$ so $\phi_F((A, I_{\mathfrak{h}}) \cdot j) = \phi_F(j) \neq 0$ as desired.

Now consider $k = (A, -I_{\mathfrak{h}}) \in K$. By the fact that $f_{s^p t^q}$ is homogeneous of degree $p+q$, $f_{s^p t^q}((A, -I_{\mathfrak{h}}) \cdot j) = (-1)^{p+q} f_{s^p t^q}(j)$. If we think of F_* as a matrix, we have that some of the rows of $F_{*(A,-I_{\mathfrak{h}})\cdot j}$ are negatives of the corresponding rows of F_{*j}. This in turn implies that some of the determinants of the $R \times R$ minors of $F_{*(A,-I_{\mathfrak{h}})\cdot j}$ may be the negative of the corresponding $R \times R$ minors of F_{*j}. However, ϕ_F calculates the sum of the squares of the determinants of these minors, so we have that $\phi_F((A, -I_{\mathfrak{h}}) \cdot j) = \phi_F(j) \neq 0$.

Next, recall

$$\phi_{T_2} = (|j(e_1)|^2 - |j(e_2)|^2) + (\langle j(e_1), j(e_2) \rangle)^2. \tag{4.30}$$

We first notice that for $m, n \in \{1,2\}$ we have $\langle j(e_m), j(e_n)\rangle = \mathrm{tr}(j(e_m)j(e_n)^*) = \mathrm{tr}((\mathrm{Ad}(A)j(e_m))(\mathrm{Ad}(A)j(e_n))^*) = \langle \mathrm{Ad}(A)j(e_m), \mathrm{Ad}(A)j(e_n)\rangle$ since $A^{-1} = A^*$. Similarly, $\langle j(e_m), j(e_n)\rangle = \langle \mathrm{Ad}(A)j(-e_m), \mathrm{Ad}(A)j(-e_n)\rangle$. Thus $\phi_{T_2}(k \cdot j) = \phi_{T_2}(j)$ for all $k = (A, \pm I_{\mathfrak{h}}) \in K$, i.e. ϕ_{T_2} is constant on the orbits of the action of K. Thus we may conclude that if $\phi_{T_2}(j) \neq 0$, $\phi_{T_2}(k \cdot j) \neq 0$ as well.

Since

$$\phi_{T_3} = \mathrm{tr}(6j(e_1)^2 j(e_2) + 2j(e_2)^3), \tag{4.31}$$

it is easy to see that $\phi_{T_3}(k \cdot j) = \pm\phi_{T_3}(j)$ for all $k \in K$. Similarly $\phi_{T_i}(k \cdot j) = \phi_{T_i}(j)$ for all $k \in K$.

Now suppose that ϕ is nonzero at j. This means that ϕ_j, the map which measures the commutativity of j, has one-dimensional kernel. Consider $(A, I_{\mathfrak{h}}) \in K$. We have

$$\phi_{(A,I_{\mathfrak{h}})\cdot j}(B) = (\mathrm{Ad}(A)j(e_1)B - B\mathrm{Ad}(A)j(e_1), \mathrm{Ad}(A)j(e_2)B - B\mathrm{Ad}(A)j(e_2)). \tag{4.32}$$

We can set up a 1-1 correspondence between the elements which commute with both $j(e_1)$ and $j(e_2)$ and the elements which commute with both $\mathrm{Ad}(A)j(e_1)$ and $\mathrm{Ad}(A)j(e_2)$. The correspondence is achieved by $B \mapsto ABA^{-1}$. We know from before that if $\phi(j) \neq 0$, then the only matrices commuting with both $j(e_1)$ and $j(e_2)$ are the scalar matrices. But if B is a scalar matrix, then $ABA^{-1} = B$. Thus the only matrices commuting with $\mathrm{Ad}(A)j(e_1)$ and $\mathrm{Ad}(A)j(e_2)$ are still just the scalar matrices. Hence if $\phi(j) \neq 0$, then $\phi((A, I_{\mathfrak{h}}) \cdot j) \neq 0$. A similar argument shows that if $\phi(j) \neq 0$ then $\phi((A, -I_{\mathfrak{h}}) \cdot j) \neq 0$ as well. We can also exploit this argument to show that $\tilde{\phi}(j) \neq 0$ implies $\tilde{\phi}(k \cdot j) \neq 0$ for any $k \in K$.

Thus we may conclude that $\mathcal{O}$ is closed under the action of K.

Now, from Section 4.3 we know that for each $j \in \mathcal{O}$, the orbit of j under the action of K is equal to E_j and that the stabilizer subgroup of j is

- $Z \ = \{(e^{i\alpha} I_n, I_{\mathfrak{h}}) | (e^{i\alpha})^n = 1\}$ for $\mathfrak{su}$(n) and
- $Z \ = \{(\pm I_{2n}, I_{\mathfrak{h}})\}$ for $\mathfrak{sp}$(n)

both of which are finite. Therefore the compact group K/Z acts freely on P_{j_0}. According to the properties of compact group actions (see [Bre72] p. 82-86), there is a submanifold N_{j_0} of P_{j_0} such that $N_{j_0} \times K/Z$ is homeomorphic to a neighborhood of j_0 in P_{j_0}. Since N_{j_0} lies in P_{j_0} each of its elements are isospectral to each other. On the other hand, since N_{j_0} is transverse to the orbits of K/Z, no two of its elements are equivalent to each other.

Finally, since Z is finite, dim$K/Z =$ K. Thus

- for $\mathfrak{su}$(n),

$$
\begin{aligned}
d &= \dim N_{j_0} && (4.33) \\
&= \dim P_{j_0} - \dim K && (4.34) \\
&= (2\dim \mathfrak{su}(\mathrm{n}) - R) - \dim \mathfrak{su}(\mathrm{n}) && (4.35) \\
&= n^2 - 1 - R && (4.36) \\
&\geq n^2 - 1 - \frac{n^2 + 3n}{2}. && (4.37)
\end{aligned}
$$

- For $\mathfrak{sp}(\mathrm{n})$,

$$
\begin{align}
d &= \dim N_{j_0} \tag{4.38}\\
&= \dim P_{j_0} - \dim K \tag{4.39}\\
&= (2\dim \mathfrak{sp}(\mathrm{n}) - R) - \dim \mathfrak{sp}(\mathrm{n}) \tag{4.40}\\
&= \dim \mathfrak{sp}(\mathrm{n}) - R \tag{4.41}\\
&\geq (2n^2 + n) - (n^2 + 2n) \tag{4.42}\\
&= n^2 - n. \tag{4.43}
\end{align}
$$

Therefore, for any $j_0 \in \mathcal{O}$, any parameterization of N_{j_0} gives us a d-parameter non-trivial isospectral deformation of j_0. $\square$

Chapter 5

Nontrivial Isospectral Deformations

In the first section of this chapter we prove a general nonisometry principle for families of metrics arising from j-maps via the construction defined in Section 3.3. In particular, letting $\mathfrak{g}_n$ denote $\mathfrak{so}$(n), $\mathfrak{su}$(n) or $\mathfrak{sp}$(n), we prove that for any element j contained in a family of generic, pairwise nonequivalent j-maps into $\mathfrak{g}_n$, there is at most one other element j' of the family such that $g_{j'}$ is isometric to g_j (except in the case of $\mathfrak{so}(8)$ where there are at most five other elements). In the second section, we complete our program by combining Theorem 3.4.4, Proposition 4.1.1, and the result of Section 5.1 to produce examples of multidimensional nontrivial isospectral deformations of metrics on $SO(n)$ ($n = 9, n \geq 11$), Spin(n) ($n = 9, n \geq 11$), $SU(n)$ ($n \geq 7$), and $Sp(n)$ ($n \geq 5$).

5.1 Isometry and Equivalence

In this chapter, G_n denotes $SO(n)$, $\mathrm{Spin}(n)$, $SU(n)$, or $Sp(n)$ and $\mathfrak{g}_n$ denotes the associated Lie algebra $\mathfrak{so}(\mathrm{n})$, $\mathfrak{su}(\mathrm{n})$, or $\mathfrak{sp}(\mathrm{n})$. Let $j : \mathfrak{h} \to \mathfrak{g}_{\mathrm{n}}$ be a linear map and let $\mathrm{I}_0(\mathrm{g_j})$ denote the identity component of the isometry group of $(\mathrm{G_{n+p}}, g_j)$. Let $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$ denote the isotropy subgroup at e of $\mathrm{I}_0(\mathrm{g_j})$. For $x \in \mathrm{G_{n+p}}$, denote left (resp. right) translation by x by L_x (resp. R_x).

Theorem 5.1.1. [OT76] *Let G be a compact, connected simple Lie group and ds^2 be a left-invariant Riemannian metric on G. Then for each isometry f contained in the identity component of the group of isometries of (G, ds^2) there exist $x, y \in G$ such that $f = L_x \circ R_y$.*

In particular, for $\alpha \in \mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$, we have that there exists some $x \in \mathrm{G_{n+p}}$ such that α is equal to conjugation of $\mathrm{G_{n+p}}$ by x. Since α fixes the identity, at the Lie algebra level we have that α_* is equal to $\mathrm{Ad}(x)$.

Proposition 5.1.2. *Suppose (M, g) and (M', g') are two Riemannian manifolds and that $\mu : (M, g) \to (M', g')$ is an isometry. Then the isometry group of (M, g) is isomorphic to the isometry group of (M', g'). Furthermore, for any point $p \in M$, the subgroup of isometries fixing p is isomorphic to the subgroup of isometries fixing $\mu(p)$.*

Proof. Define a map from the isometry group of (M, g) to the isometry group of (M', g') by taking each isometry α of (M, g) to the isometry $\mu\alpha\mu^{-1}$ of (M', g'). It is easy to see that this is an isomorphism.

Now suppose that α is in the isotropy subgroup of the point $p \in M$. Then

$$\mu\alpha\mu^{-1}(\mu(p)) = \mu\alpha(p) = \mu(p) \tag{5.1}$$

Hence $\mu\alpha\mu^{-1}$ fixes $\mu(p)$ so the isomorphism defined above restricts to an isomorphism of the isotropy subgroups. $\square$

Corollary 5.1.3. *Given two j-maps $j, j' : \mathfrak{h} \to \mathfrak{g}_{\mathrm{n}}$, suppose there exists an isometry $\mu : (\mathrm{G}_{\mathrm{n+p}}, g_j) \to (\mathrm{G}_{\mathrm{n+p}}, g_{j'})$. Then $\mathrm{I}_0(\mathrm{g_j})$ is isomorphic to $\mathrm{I}_0(\mathrm{g_{j'}})$ and $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$ is isomorphic to $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_{j'}})$.*

Proof. First compose μ with $L_{\mu(e)^{-1}}$. Then we have an isometry from $(\mathrm{G}_{\mathrm{n+p}}, g_j)$ to $(\mathrm{G}_{\mathrm{n+p}}, g_{j'})$ which carries e to e. The isomorphism defined in Proposition 5.1.2 carries the identity component of the isometry group of $(\mathrm{G}_{\mathrm{n+p}}, g_j)$ to the identity component of the isometry group of $(\mathrm{G}_{\mathrm{n+p}}, g_{j'})$ so the corollary follows. $\square$

Proposition 5.1.2 implies the following.

Proposition 5.1.4. *Suppose G is a compact simple group with left-invariant metrics g and g' neither of which is bi-invariant. If $\mu : (G, g) \to (G, g')$ is an isometry, then (after possibly composing with $L_{\mu(e)^{-1}}$), μ is an automorphism of G.*

Proof. If $\mu(e) \neq e$, compose μ with $L_{\mu(e)^{-1}}$ so that the isometry carries e to itself. Since G is compact, the isometry groups of (G, g) and (G, g') are also compact. Thus we may write the isometry group of (G, g) as $G_1 \times G_2 \times \cdots \times G_s \times T/Z$ and the isometry group of (G, g') as $G_1' \times G_2' \times \cdots \times G_t' \times T'/Z'$ where each $G_i^{(\prime)}$ is simple, T' is a torus, and $Z^{(\prime)}$ is central. Each isometry groups contains a copy of G in the form of left translations. Furthermore, since neither g nor g' is bi-invariant, each isometry group contains exactly one copy of G. By Proposition 5.1.2, we know that these isometry groups must be isomorphic via conjugation by μ. Any isomorphism from the isometry group of (G, g) to the isometry group of (G, g') must carry simple factors to simple factors. Since G is the only simple factor of its dimension, we have that any isomorphism carries G to G. In our particular case, this means that for every $x \in G$ there exists $x' \in G$ such that $\mu L_x \mu^{-1} = L_{x'}$. Thus for each $x, y \in G$,

$$\begin{aligned}\mu(xy) &= \mu(L_{xy})(e) = \mu(L_x L_y)(e) = \mu L_x \mu^{-1} \mu L_y(e) \\ &= L_{x'} \mu L_y(e) = L_{x'} L_{y'} \mu(e) = L_{x'} L_{y'}(e) = x'y' = \mu(x)\mu(y). \quad (5.2)\end{aligned}$$

□

Corollary 5.1.5. *Given two nonzero j-maps, j and j', suppose there exists an isometry $\mu : (\mathrm{G_{n+p}}, g_j) \to (\mathrm{G_{n+p}}, g_{j'})$. Then (after possibly composing μ with $L_{\mu(e)^{-1}}$) μ is an automorphism of* $\mathrm{G_{n+p}}$.

Proof. Each of $SO(n)$, $\mathrm{Spin}(n)$, $SU(n)$, and $Sp(n)$ is compact and simple. Since neither j nor j' is trivial, we have that neither g_j nor $g_{j'}$ is bi-invariant. The corollary then follows directly from Proposition 5.1.4. □

Lemma 5.1.6. *Let $j, j' : \mathfrak{h} \to \mathfrak{g}_n$ be linear maps. Suppose there exists an isometry $\mu : (\mathrm{G_{n+p}}, g_j) \to (\mathrm{G_{n+p}}, g_{j'})$ such that $\mu(T^2) = T^2$. Then there is an element $C \in O(\mathfrak{h})$ such that $j(z) = \mu_*^{-1} j'(Cz)$ for all $z \in \mathfrak{h}$.*

Proof. Without loss of generality, assume $\mu(e) = e$. If μ maps T^2 to itself, then it must isometrically map the Lie algebra $\mathfrak{h}$ to itself. This implies that there is an element $C \in O(\mathfrak{h})$ such that μ_* restricted to $\mathfrak{h}$ is equal to C.

From Corollary 5.1.5, we know that if μ is an isometry, it is also an automorphism of $\mathrm{G_{n+p}}$. Thus, if μ maps T^2 to itself in $\mathrm{G_{n+p}}$, it must also isomorphically map the identity component of the centralizer of T^2 in $\mathrm{G_{n+p}}$ to itself. At the Lie algebra level, direct calculation shows that the centralizer of $\mathfrak{h}$ in

- $\mathfrak{so}(\mathrm{n}+4)$ is $\mathfrak{so}(\mathrm{n}) \oplus \mathfrak{h}$. Thus the identity component of the centralizer of T^2 in $SO(n+4)$ is $SO(n) \times T^2$ and the identity component of the centralizer of T^2 in $\mathrm{Spin}(n+4)$ is $\mathrm{Spin}(n) \times T^2$.

- $\mathfrak{su}(\mathrm{n}+3)$ is $\mathfrak{su}(\mathrm{n}) \oplus tu \oplus \mathfrak{h}$ where

$$u = \begin{bmatrix} i/n & & & & & \\ & \ddots & & & & \\ & & i/n & & & \\ & & & -i/3 & & \\ & & & & -i/3 & \\ & & & & & -i/3 \end{bmatrix} \in \mathfrak{su}(\mathrm{n}+3)\,. \tag{5.3}$$

  All three of $\mathfrak{su}(\mathrm{n})$, u, and $\mathfrak{h}$ commute with each other so this is a Lie algebra direct sum. Letting U denote the one-parameter subgroup associated to u, we have the identity component of the centralizer of T^2 in $\mathrm{SU}(\mathrm{n}+3)$ is $SU(n) \times U \times T^2$.

- $\mathfrak{sp}(\mathrm{n}+2)$ is $\mathfrak{sp}(\mathrm{n}) \oplus \mathfrak{h}$. Thus the identity component of the centralizer of T^2 in $Sp(n+2)$ is $Sp(n) \times T^2$.

In each case, the identity component of the centralizer of T^2 is the product of a simple group, G_n, with a torus. Therefore $\mu(G_n) = G_n$ and μ_* is a Lie algebra automorphism of $\mathfrak{g}_n$. For any $X \in \mathfrak{g}_\mathrm{n}$, we have that $X - j^t(X) \in \mathfrak{h}^{\perp_{g_j}}$ is mapped to $\mu_* X - Cj^t(X)$.

On the other hand, since $\mu_* X \in \mathfrak{g}_\mathrm{n}$ and $Cj^t(X) \in \mathfrak{h}$ and since $\mu_* : \mathfrak{h}^{\perp_{g_j}} \to \mathfrak{h}^{\perp_{g_{j'}}}$, it must be the case that $Cj^t(X) = j'^t(\mu_* X)$ for all $X \in \mathfrak{g}_\mathrm{n}$. Otherwise, $Cj^t(X) = j'^t(\mu_* X) + Z$ for some nonzero $Z \in \mathfrak{h}$ depending on X. But in this case, $\mu_* X - Cj^t(X) = \mu_* X - j'^t(\mu_* X) - Z$ which is not in $\mathfrak{h}^{\perp_{g_{j'}}}$.

Finally, taking transposes, we see that the condition $j^t(X) = C^{-1} j'^t(\mu_* X)$ for all $X \in \mathfrak{g}_\mathrm{n}$ implies $j(z) = \mu_*^{-1} j'(Cz)$ for all $z \in \mathfrak{h}$. □

Remark 5.1.7. From the proof of Lemma 5.1.6, we saw that under the hypothesis of the lemma, μ is an automorphism of G_n. Suppose μ restricted to G_n is an inner automorphism. That is, suppose μ restricted to G_n equals conjugation by an element

A of G_n. In this case μ_* restricted to $\mathfrak{g}_n$ is equal to $\mathrm{Ad}(A)$. But then by the proof, $j(z) = \mathrm{Ad}(A^{-1})\, j'(Cz)$ for all $z \in \mathfrak{h}$. In other words, j and j' are equivalent.

Genericity Condition 5.1.8. We say that $j : \mathfrak{h} \to \mathfrak{g}_n$ is generic if there are only finitely many $A \in G_n$ such that $j(z) = \mathrm{Ad}(A)j(z)$ for all $z \in \mathfrak{h}$.

Let $C(G_n)$ denote the center of G_n. From [GW97] and the proof of Propostion 4.1.1 we know that for $G_n = SO(n)$, $SU(n)$, or $Sp(n)$ and $j \in \mathcal{O}$, the group action of $G_n \times O(\mathfrak{h})$ on the space L of j-maps given by $((A, C) \cdot j)(z) = \mathrm{Ad}(A)j(C^{-1}(z))$ has stability subgroup $C(G_n) \times \{I_{\mathfrak{h}}\}$. But $C(G_n)$ is finite so we conclude that each $j \in \mathcal{O}$ is generic. On the other hand, let $A \in SO(n)$ and let $\overline{A}$ be a lift of A in $\mathrm{Spin}(n)$. We know that $\mathrm{Ad}(\overline{A}) = \mathrm{Ad}(A)$. But $\mathrm{Spin}(n)$ is a two-fold cover of $SO(n)$. Therefore for any $j : \mathfrak{h} \to \mathfrak{so}(\mathrm{n})$ arising from Proposition 4.1.1 we have a finite number of elements $\overline{A}$ such that $j(z) = \mathrm{Ad}(\overline{A})j(z)$. In other words, j is generic with respect to $\mathrm{Spin}(n)$.

Lemma 5.1.9. *Let $j : \mathfrak{h} \to \mathfrak{g}_{\mathrm{n}}$ be generic and let g_j be the associated metric on $\mathrm{G}_{\mathrm{n+p}}$. For $\mathrm{G}_{\mathrm{n+p}}$ equal to $SO(n+4)$, $\mathrm{Spin}(n+4)$, or $Sp(n+2)$, let D be the group of isometries of $(\mathrm{G}_{\mathrm{n+p}}, g_j)$ generated by the set $\{L_x \circ R_{x^{-1}} | x \in T^2\}$. For $\mathrm{G}_{\mathrm{n+p}}$ equal to $SU(n+3)$, let D be the group of isometries of $(\mathrm{G}_{\mathrm{n+p}}, g_j)$ generated by the set $\{L_x \circ R_{x^{-1}} | x \in U \times T^2\}$, where U is as in the proof of Lemma 5.1.6. Then D is a maximal torus in $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$.*

Proof. First we check that $D \subset \mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$. Let $\mu \in D$. Direct calculation shows that when μ_* acts on $\mathfrak{g}_{\mathrm{n+p}}$ it fixes pointwise $\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h}$ and sends $(\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}}$ to itself. Thus for $X, Y \in \mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h}$,

$$g_j(\mu_* X, \mu_* Y) = g_j(X, Y). \tag{5.4}$$

Furthermore, by Section 3.3, g_j restricted to $(\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}}$ is equal to g_0. For $X, Y \in$

$(\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}}$,

$$\begin{aligned} g_j(\mu_* X, \mu_* Y) &= g_0(\mu_* X, \mu_* Y) && (5.5) \\ &= g_0(X, Y) && (5.6) \end{aligned}$$

since g_0 is by bi-invariant.

Finally, if $X \in \mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h}$ and $Y \in (\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}}$ then since $(\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}} = (\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_\mathrm{j}}}$ and $\mu_* Y \in (\mathfrak{g}_{\mathrm{n}} \oplus \mathfrak{h})^{\perp_{\mathrm{g}_0}}$,

$$g_j(\mu_* X, \mu_* Y) = g_j(X, \mu_* Y) = 0. \tag{5.7}$$

Thus μ acts isometrically on g_j so $D \subset \mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$.

Now recall that every element of $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$ is of the form $L_x \circ R_{x^{-1}}$ for some $x \in \mathrm{G_{n+p}}$. Let $C(\mathrm{G_{n+p}})$ denote the finite center of $\mathrm{G_{n+p}}$. We identify $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$ with a subgroup of $\mathrm{G_{n+p}}/C(\mathrm{G_{n+p}})$ via the map which sends $L_x \circ R_{x^{-1}}$ to the coset of x in $\mathrm{G_{n+p}}/C(\mathrm{G_{n+p}})$. Under this correspondence, we consider D a subgroup of $\mathrm{G_{n+p}}/C(\mathrm{G_{n+p}})$. If $L_\alpha \circ R_{\alpha^{-1}} \in \mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$ commutes with D, then under the identification of D with a subgroup of $\mathrm{G_{n+p}}/C(\mathrm{G_{n+p}})$, for each $x \in T^2$ (resp. $U \times T^2$) $\alpha x \alpha^{-1} = xz$ for some $z \in C(\mathrm{G_{n+p}})$. Elements of $C(\mathrm{G_{n+p}})$ are scalar matrices. Since conjugation preserves eigenvalues it must be the case that $\alpha x \alpha^{-1} = x$. This implies that when α acts by isometry (i.e. conjugation) on $\mathrm{G_{n+p}}$, it fixes T^2 pointwise. Thus α is in the centralizer of T^2 in $\mathrm{G_{n+p}}$.

From the proof of Lemma 5.1.6 we know that the identity component of the centralizer of T^2 in

- $SO(n+4)$ is $SO(n) \times T^2$ and the identity component of the centralizer of T^2 in $\mathrm{Spin}(n+4)$ is $\mathrm{Spin}(n) \times T^2$.

- SU(n + 3) is $SU(n) \times U \times T^2$ where U is as in the proof of Lemma 5.1.6.

- $Sp(n+2)$ is $Sp(n) \times T^2$.

First suppose α is in the identity component of the centralizer of T^2. Then for $SO(n+4)$, Spin$(n+4)$, and $Sp(n+2)$, α is equal to a product AZ and for $SU(n+3)$, α equals AUZ for some $A \in G_n$, $Z \in T^2$. In this case, $(L_\alpha \circ R_{\alpha^{-1}})_*$ restricted to $\mathfrak{g}_n$ equals $\mathrm{Ad}(A)$ and by the proof of Lemma 5.1.6

$$j(z) = \mathrm{Ad}(A) j(z) \text{ for all } z \in \mathfrak{h}\,. \tag{5.8}$$

But by the genericity of j, there are only finitely many A for which Equation 5.8 holds and thus only finitely many A such that $\alpha = AZ$ (resp. AUZ) for some $Z \in T^2$.

On the other hand, if α is not an element of the identity component of the centralizer of T^2 then $\alpha = AZP$ (resp. $AUZP$) for some element P of a discrete set contained in the centralizer of T^2. In any case, we have now shown that if α centralizes T^2, then it must be contained in a subgroup of $\mathrm{G_{n+p}}$ composed of a discrete number of copies of T^2 (resp. $U \times T^2$). Hence $L_\alpha \circ R_{\alpha^{-1}}$ is contained in a subgroup of $\mathrm{I_0^e(g_j)}$ which is composed of a discrete number of copies of D. In other words, D is not contained in a higher dimensional connected torus and hence is a maximal torus in $\mathrm{I_0^e(g_j)}$. ☐

Theorem 5.1.10. *Let j and j' be generic linear maps. Suppose that $\mu : (\mathrm{G_{n+p}}, g_j) \to (\mathrm{G_{n+p}}, g_{j'})$ is an isometry. Then there exists an element $C \in O(\mathfrak{h})$ such that $j(z) = \mu_*^{-1} j'(Cz)$ for all $z \in \mathfrak{h}$. By Remark 5.1.7, if μ restricted to G_n is an inner automorphism, then j and j' are equivalent.*

Proof. Suppose that $\mu : (\mathrm{G_{n+p}}, g_j) \to (\mathrm{G_{n+p}}, g_{j'})$ is an isometry. We may assume that $\mu(e) = e$. By Lemma 5.1.6, it suffices to show that $\mu(T^2) = T^2$.

By Corollary 5.1.3, we have that $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$ is isomorphic to $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_{j'}})$ via the map which carries $\alpha \in \mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$ to $\mu\alpha\mu^{-1} \in \mathrm{I}_0^{\mathrm{e}}(\mathrm{g_{j'}})$. According to Lemma 5.1.9, D is a maximal torus in $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_j})$ and so the isomorphism carries D to a maximal torus in $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_{j'}})$. All maximal tori in a compact Lie group are conjugate so, after possibly composing μ with an element of $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_{j'}})$, we may assume that conjugation by μ carries D to the similarly defined set in $\mathrm{I}_0^{\mathrm{e}}(\mathrm{g_{j'}})$.

- For $SO(n+4)$, $\mathrm{Spin}(n+4)$, and $Sp(n+2)$, this implies that for any $a \in T^2$, $\mu \circ L_a \circ R_{a^{-1}} \circ \mu^{-1} = L_b \circ R_{b^{-1}}$ for some $b \in T^2$. On the other hand, by Corollary 5.1.5, we know μ is an automorphism of $\mathrm{G_{n+p}}$. Thus, for any $x \in \mathrm{G_{n+p}}$,

  $$\mu \circ L_a \circ R_{a^{-1}} \circ \mu^{-1}(x) = \mu(a\mu^{-1}(x)a^{-1}) = \mu(a)x\mu^{-1}(a) = L_{\mu(a)} \circ R_{\mu(a^{-1})}(x).$$

  In other words, $\mu(a) = bz$ for some $z \in C(\mathrm{G_{n+p}})$. For each of $SO(n+4)$, $\mathrm{Spin}(n+4)$, and $Sp(n+2)$, $C(\mathrm{G_{n+p}})$ is finite. Since μ is continuous and since $\mu(e) = e$, we have that $\mu(T^2) = T^2$. Thus $j(z) = \mu_*^{-1} j'(Cz)$ for all $z \in \mathfrak{h}$.

- For $SU(n+3)$, we have that for any $a \in U \times T^2$, $\mu \circ L_a \circ R_{a^{-1}} \circ \mu^{-1} = L_b \circ R_{b^{-1}}$ for some $b \in U \times T^2$.

  Alternatively, for any $x \in \mathrm{G_{n+p}}$ since μ is an automorphism,

  $$\mu \circ L_a \circ R_{a^{-1}} \circ \mu^{-1}(x) = \mu(a\mu^{-1}(x)a^{-1}) = \mu(a)x\mu^{-1}(a) = L_{\mu(a)} \circ R_{\mu(a^{-1})}(x).$$

  In other words, $\mu(a) = bz$ for some $z \in C(SU(n+3))$. But $C(SU(n+3))$ is finite. Since μ is continuous and since $\mu(e) = e$, we conclude that $\mu(U \times T^2) = U \times T^2$.

  Since μ maps $U \times T^2$ to $U \times T^2$, at the Lie algebra level, μ_* maps $tu \oplus \mathfrak{h}$ to $tu \oplus \mathfrak{h}$. Now consider μ as an automorphism. The automorphism group of $\mathrm{SU(n+3)}$

is generated by the inner automorphisms and one outer automorphism, namely complex conjugation.

At the Lie algebra level, conjugating an element $X \in \mathfrak{su}(\mathrm{n}+3)$ by any element of $\mathrm{SU}(\mathrm{n}+3)$ preserves the eigenvalues of X. In particular, u has eigenvalue i/n with multiplicity n and eigenvalue $-i/3$ with multiplicity 3. No other element of $tu \oplus \mathfrak{h}$ has the same eigenvalues. Therefore each inner automorphism of $\mathfrak{su}(\mathrm{n}+3)$ maps u to u.

Similarly, at the Lie algebra level, the outer automorphism of $\mathrm{SU}(\mathrm{n}+3)$ negates the eigenvalues of $X \in \mathfrak{su}(\mathrm{n}+3)$. For each $t \in \mathbb{R}$, this sends tu to $-tu$. Thus the vector space spanned by u is fixed.

Since μ is an isometry, μ_* maps the g_j-orthogonal complement of the space spanned by u to the $g_{j'}$-orthogonal complement of the space spanned by u. But $tu \oplus \mathfrak{h}$ is contained in $\mathfrak{g}_{\mathrm{n}}^{\perp_{g_0}}$ so both g_j and $g_{j'}$ restricted to $tu \oplus \mathfrak{h}$ are equal to the bi-invariant metric g_0. The inner product g_0 is given by $g_0(c, d) = \mathrm{tr}(cd^*)$. It is easy to see that $\mathfrak{h}$ is g_0-orthogonal to u and therefore $\mu(T^2) = T^2$. By Lemma 5.1.6, $j(z) = \mu_*^{-1} j'(Cz)$ for all $z \in \mathfrak{h}$.

$\square$

Theorem 5.1.11. *Suppose $j_0 : \mathfrak{h} \to \mathfrak{g}_n$ is contained in a family of generic linear maps which are pairwise nonequivalent. For $\mathfrak{so}(\mathrm{n})$ (n odd and $n \geq 5$) and $\mathfrak{sp}(\mathrm{n})$ ($n \geq 3$) there is no other map j contained in the family such that g_{j_0} and g_j are isometric. For $\mathfrak{so}(\mathrm{n})$ (n even and $n \geq 10$) and $\mathfrak{su}(\mathrm{n})$ ($n \geq 2$) there is at most one other linear map j in the family such that g_{j_0} and g_j are isometric. For $\mathfrak{so}(8)$ there are at most five other maps.*

Proof. First consider the automorphism groups of $\mathfrak{so}(\mathrm{n})$, $\mathfrak{su}(\mathrm{n})$, and $\mathfrak{sp}(\mathrm{n})$.

- $\mathfrak{so}$(n), (n odd, $n \geq 5$): All automorphisms are equal to Ad(A) for some $A \in SO(n)$.

- $\mathfrak{so}$(n), ($n = 8$): The automorphism group is generated by three types of automorphisms: Ad(A) where $A \in SO(n)$, Ad(B) where

$$B = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I_{n-2} \end{bmatrix} \in O(n), \tag{5.9}$$

  and an outer automorphism which is not given by conjugation. The factor group, $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}^0(\mathfrak{g})$ of automorphisms modulo inner automorphism is isomorphic to the symmetric group on three letters.

- $\mathfrak{so}$(n), (n even, $n \geq 10$): All automorphisms are generated by Ad(A) where $A \in SO(n)$ or Ad(B) where $B \in O(n)$ as above.

- $\mathfrak{su}$(n) ($n \geq 2$): The automorphisms are generated by the inner automorphisms Ad(A) where $A \in SU(n)$, and the outer automorphism which takes an element to its complex conjugate.

- $\mathfrak{sp}$(n) ($n \geq 3$): All automorphism are inner.

Suppose j and j' are two linear maps such that g_j and $g_{j'}$ are both isometric to g_{j_0}. From Theorem 5.1.10 we have Lie algebra automorphisms ϕ, ϕ' of $\mathfrak{g}_n$ and elements C, C' of $O(\mathfrak{h})$ such that

$$j(z) = \phi j_0(Cz) \tag{5.10}$$

and

$$j'(z) = \phi' j_0(C'z) \tag{5.11}$$

for all $z \in \mathfrak{h}$.

If ϕ and ϕ' are in the same coset of $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}^0(\mathfrak{g})$, then they differ by an inner automorphism. But in this case j and j' are equivalent. Since our family consists of pairwise nonequivalent linear maps, j is equal to j'. The theorem now follows from the fact that for $\mathfrak{so}(\mathrm{n})$ (n odd, $n \geq 5$), $\mathfrak{su}(\mathrm{n})$ ($n \geq 2$), and $\mathfrak{sp}(\mathrm{n})$ ($n \geq 3$), $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}^0(\mathfrak{g})$ has one element, for $\mathfrak{so}(\mathrm{n})$ (n even, $n \geq 10$), $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}^0(\mathfrak{g})$ has two elements, and for $\mathfrak{so}(8)$, $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}^0(\mathfrak{g})$ has six elements. $\square$

5.2 Examples

In the final section of this paper we combine our previous results to conclude the existence of new nontrivial multiparameter isospectral deformations of metrics on all of the classical compact simple Lie groups.

Example 5.2.1. Recall that given two linear maps $j, j' : \mathfrak{h} \to \mathfrak{g}_n$, when we construct metrics g_j and $g_{j'}$ on G_{n+p} according to the construction given in Section 3.3, Theorem 3.4.4 says that if j and j' are isospectral then the metrics g_j and $g_{j'}$ are isospectral. Furthermore, Proposition 4.1.1 gives the existence of a Zariski open set $\mathcal{O}$ contained in the set L of all j-maps into $\mathfrak{g}_n$ such that each $j_0 \in \mathcal{O}$ is contained in a d-parameter family $\mathcal{F}$ of j-maps which are isospectral but not equivalent. Here, d depends on $\mathfrak{g}_n$ as follows:

$\mathfrak{g}_n$	d
$\mathfrak{so}$(n)	$d \geq n(n-1)/2 - [\frac{n}{2}]([\frac{n}{2}]+2)$
$\mathfrak{su}$(n)	$d \geq n^2 - 1 - \frac{n^2+3n}{2}$
$\mathfrak{sp}$(n)	$d \geq n^2 - n$

By the construction of $\mathcal{O}$, we know that each element of $\mathcal{F}$ is generic.

First consider the case of $\mathfrak{so}$(n). Recall from Remark 4.1.2 that Theorem 4.1.1 was originally proven for $\mathfrak{so}$(n) with associated Lie group $O(n)$. Thus Theorem 4.1.1 tells us that there exists a continuous family of elements $a_{zp} \in O(n)$ such that if $j_p \in \mathcal{F}$, $j_0(z) = Ad(a_{zp})j_p(z)$. But since we may choose $a_{z0} = I_n$ and the family is continuous, we conclude that we may choose $a_{zp} \in SO(n)$ for all p. Thus the families are isospectral with respect to $SO(n)$.

Since the automorphism group of $\mathfrak{so}$(n) ($n = 5, 7$, and $n \geq 9$) is contained in $\{\mathrm{Ad}(A) | A \in O(n)\}$, we have from the proof of Theorem 5.1.11 that no two elements of $\mathcal{F}$ give rise to isometric metrics. For $\mathfrak{so}(8)$, there is one generator of the automorphism group which is not given by $\mathrm{Ad}(A)$ for some $A \in O(n)$. Thus for any element of $\mathcal{F}$ there is at most one other element which could give rise to an isometric metric. For fixed $j_0 \in \mathcal{O}$, we may choose $\mathcal{F}$ small enough that no other element of $\mathcal{F}$ produces a metric isometric to g_{j_0}, thereby obtaining an isospectral deformation of g_{j_0}.

Now, for $A \in SO(n)$, let $\overline{A}$ be a lift of A in Spin(n). Then the map $\mathrm{Ad}(\overline{A}) : \mathfrak{so}(\mathrm{n}) \to \mathfrak{so}(\mathrm{n})$ is equal to the map $\mathrm{Ad}(A) : \mathfrak{so}(\mathrm{n}) \to \mathfrak{so}(\mathrm{n})$. Thus the orbits of Ad(Spin(n)) in $\mathfrak{so}$(n) are equal to the orbits of $\mathrm{Ad}(SO(n))$. Therefore if j and j' are isospectral with respect to $SO(n)$, they are also isospectral with respect to Spin(n). Fix $j_0 \in \mathcal{O}$ and consider the metric g_{j_0} on Spin(n). By an argument similar to the one for $SO(n)$, we conclude that for $n = 5, 7$ or $n \geq 9$, there exists a nontrivial isospectral deformation of g_{j_0} such that no two metrics in the deformation are isometric to each other. For $n = 8$ there exists a multiparameter nontrivial isospectral deformation of g_{j_0} such that for any metric in the deformation there is at most one other metric isospectral to it.

Next suppose $\mathcal{F}$ is a continuous d-parameter family of isospectral, nonequivalent j-maps into $\mathfrak{su}$(n). Fix $j_0 \in \mathcal{F}$. For $n \geq 4$, Theorem 5.1.11 tells us that we may pick $\mathcal{F}$ small enough so that there is no other element of $\mathcal{F}$ which gives a metric isometric to g_{j_0}. Thus we have a d-parameter isospectral deformations of metrics on $SU(n)$ for $n \geq 7$. Furthermore, for any metric within the deformation, there is at most one other isometric metric contained in the deformation. For n greater than 7, d is greater than 1.

Finally, a similar argument shows that for $n \geq 5$, we have multiparameter isospectral deformations of metrics on $Sp(n)$ such that no two metrics in a given deformation are isometric.

Thus we have produced isospectral deformations of metrics on each of $SO(n)$ $(n = 9, n \geq 11)$, Spin(n) $(n = 9, n \geq 11)$, $SU(n)$ $(n \geq 7)$, and $Sp(n)$ $(n \geq 5)$.

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