THE SPECTRA OF DIGRAPHS WITH MORITA EQUIVALENT C^*-ALGEBRAS

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Abstract. Eilers et al. have recently completed the geometric classification of unital graph C^*-algebras up to Morita equivalence using a set of moves on the corresponding digraphs. We explore the question of whether these moves preserve the nonzero elements of the spectrum of a finite digraph, which in this paper is allowed to have loops and parallel edges. We consider several different digraph spectra that have been studied in the literature, answering this question for the Laplace and adjacency spectra, their skew counterparts, the symmetric adjacency spectrum, the adjacency spectrum of the line digraph, the Hermitian adjacency spectrum, and the normalized Laplacian, considering in most cases two ways that these spectra can be defined in the presence of parallel edges. We show that the adjacency spectra of the digraph and line digraph are preserved by a subset of the moves, and the skew adjacency and Laplace spectra are preserved by the Cuntz splice. We give counterexamples to show that the other spectra are not preserved by the remaining moves. The same results hold if one restricts to the class of strongly connected digraphs.

Contents

1. Introduction

Given a graph, undirected or directed, there are a number of ways to associate a matrix to the graph and, as a consequence, various eigenvalue spectra that can be considered. These spectra are graph isomorphism invariants and have been studied in their own right. For example, the spectrum of the adjacency matrix of a digraph characterizes the number of cycles in the graph; see [4], [44]. See [9], [5] for basic introductions to the subject.

1991 *Mathematics Subject Classification.* Primary 05C50, 05C20; Secondary 46L35.

Key words and phrases. digraph, pseudodigraph, digraph spectrum, Morita equivalence, digraph Laplacian, skew spectrum, Hermitian spectrum, normalized Laplacian.

Beyond the study of intrinsic properties of graphs, spectra of graphs arise in the study of other mathematical structures. For instance, the spectrum of the Laplacian on a domain or manifold is a classical geometric invariant that has been extensively studied for decades. The notion of discretizing the Laplace spectrum of a manifold has led to the investigation of approximating graphs whose spectra tends to the spectrum of the manifold Laplacian. For example, by embedding a finite graph with a given Laplace spectrum into a manifold and extending the metric globally, Colin de Verdière showed that for any closed manifold M, one can choose a metric on M in such a way as to prescribe the first N eigenvalues of the Laplacian acting on functions on M; see [12, 30, 17], and [24] for the case of orbifolds.

As another important instance of associating a mathematical structure to a graph, given a directed graph $D = (V, E, r, s)$ (which in this context is often simply called a graph), we can generate a C^*-algebra $C^*(D)$ based on the information encoded in the digraph. For example, starting from either a one-vertex digraph having N loops, or from the complete digraph on N vertices, we can generate a C^*-algebra that is canonically isomorphic to the Cuntz algebra $\mathcal{O}_N$. More generally, given an irreducible matrix $\mathbf{A}_{\mathrm{b}}$ of 0's and 1's, we can interpret $\mathbf{A}_{\mathrm{b}}$ as the adjacency matrix of a digraph $D_{\mathbf{A}_{\mathrm{b}}}$. The C^*-algebra $C^*(D_{\mathbf{A}_{\mathrm{b}}})$ associated to $D_{\mathbf{A}_{\mathrm{b}}}$ is canonically isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{\mathbf{A}_{\mathrm{b}}}$. See [14] and [13].

This paper focuses on the relationship between the spectra of a finite digraph (i.e. a directed graph with finitely many vertices and edges, which we allow to have loops and parallel edges, sometimes referred to as a *pseudodigraph*) and the Morita equivalence class of its associated C^*-algebra. The main question that we address is: given a digraph, to what extent does the Morita equivalence class of the associated C^*-algebra determine the spectrum of the graph? We note that there are numerous matrices, and thus numerous eigenvalue spectra, that one can associate to a given digraph. We examine a wide collection here; see Definition 2.8.

Our work is based on recent results on a classification of graph C^*-algebras via K-theoretic and combinatorial invariants. In particular, in a series of papers [40, 42, 18, 22, 20, 21] Eilers, et al. obtained a geometric classification of unital graph C^*-algebras. Observe that a graph C^*-algebra is unital if and only if its corresponding graph has finitely many vertices (see [21, Section 2.3]). See [21] for the culmination of this work. The authors of these papers defined six "moves", labeled (S), (R), (O), (I), (C), and (P), that can be performed on digraphs having countably many vertices and edges. Although these moves can be defined for countable graphs, for their classification results, Eilers, et al. restrict their attention to graphs having finitely many vertices (see [21, Theorem 3.1 and Corollaries 3.2 and 3.3]). In [21], they show that the graph C^*-algebras $C^*(D_1)$ and $C^*(D_2)$ associated to two digraphs D_1 and D_2 are stably isomorphic if and only if D_1 and D_2 differ by a finite sequence of these six moves and their inverses. In this context, because $C^*(D_1)$ and $C^*(D_2)$ are unital C^*-algebras, they are stably isomorphic if and only if they are Morita equivalent. We note further that by [1], for a digraph D, the graph C^*-algebra $C^*(D)$ is isomorphic to $C^*(\mathcal{L}(D))$, where $\mathcal{L}(D)$ denotes the line digraph, also known as the dual graph, associated to D (see Definition 2.4).

A natural question arises from the above results: How does the spectrum of a digraph behave under the above moves? For instance, the fact that the C^*-algebras of D and $\mathcal{L}(D)$ are isomorphic, together with the fact that the nonzero elements of the adjacency spectra of D and $\mathcal{L}(D)$ coincide (see [5, p. 2183], [34, Theorem 1.4.4], and Proposition 3.4 below), indicate a potential connection. In this paper, we restrict our attention to finite graphs, i.e. graphs having finitely many vertices and edges, so that the spectrum is defined. For graphs having finitely many vertices, this amounts to the requirement that the graphs have no infinite emitters. We also consider this question with the additional restriction that the graphs are strongly connected.

Our main result is the following, corresponding to Propositions 4.2, 4.3, 4.4, and 4.5, as well as Tables 1 and 2 in Section 5.

Theorem 1.1. *Let D be a finite digraph, and consider the Moves (S), (R), (O), (I), (C), and (P) that preserve the Morita equivalence class of the C^*-algebra of D.*

(i) Moves (S), (O), and (I) preserve the multiset of nonzero elements of the adjacency spectrum $\mathrm{Spec}_{\mathbf{A}}(D)$ *of D while Moves (R), (C), and (P) do not;*

(ii) Moves (S), (O), and (I) preserve the multiset of nonzero elements of the line adjacency spectrum $\mathrm{Spec}_{\mathbf{L}}(D)$ *while Moves (R), (C), and (P) do not;*

(iii) Move (S) preserves the multiset of nonzero elements of the binary adjacency spectrum $\mathrm{Spec}_{\mathbf{A}_\mathrm{b}}(D)$ *while Moves (R), (O), (I), (C), and (P) do not; and*

(iv) Move (C) preserves the multisets of nonzero elements of the skew adjacency spectrum $\mathrm{Spec}_{\mathbf{S}}(D)$*, the binary skew adjacency spectrum* $\mathrm{Spec}_{\mathbf{S}_\mathrm{b}}(D)$*, the skew Laplace spectrum* $\mathrm{Spec}_{\Delta_S}(D)$*, and the binary skew Laplace spectrum* $\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D)$*, while Moves (S), (R), (O), (I), and (P) do not.*

The (nonzero elements of the) other spectra $\mathrm{Spec}_{\Delta}(D)$*,* $\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D)$*,* $\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_\mathrm{b}}(D)$*,* $\mathrm{Spec}_{\mathbf{H}}(D)$*, are not preserved by any of the moves.*

If D is a strongly connected digraph, then only the Moves (R), (O), (I), and (C) can be applied to D, and these claims remain true. Furthermore, the spectra $\mathrm{Spec}_{\Delta_N}(D)$*,* $\mathrm{Spec}_{\Delta_{\mathrm{b},N}}(D)$*,* $\mathrm{Spec}_{\Delta_C}(D)$*, and* $\mathrm{Spec}_{\Delta_{\mathrm{b},C}}(D)$*, which are only defined for strongly connected digraphs, are not preserved by any of these four moves.*

Note that if a move preserves a spectrum, then it is clear that the inverse of the move also preserves the spectrum.

The proofs of parts (i), (ii), and (iii) of Theorem 1.1 rely on the characterization of the adjacency spectrum and line adjacency spectrum of a digraph in terms of the number of cycles of a given length that the graph contains. The key idea of each proof consists of showing that the number of cycles of a given length is preserved by the appropriate moves. Part (iv) follows in a straightforward way from the definition of Move (C). We provide counterexamples to show that none of the other spectra are preserved. Because of the number of spectra and moves considered, Tables 1 and 2 summarize the result or counterexample that proves each case of Theorem 1.1. Observe that parts (i) and (ii) of Theorem 1.1, are consistent with the observation above that for a given digraph D, $C^*(D)$ coincides with $C^*(\mathcal{L}(D))$.

In recent years, Kumijan and Pask developed a theory of higher-rank graphs, also known as k-graphs, which provide a generalization of digraphs to higher dimensions; see [33]. Following the work of Eilers, et al., in [16] steps are taken toward extending the geometric classification of digraph C^*-algebras and their Morita invariant moves to the setting of higher-rank graphs by introducing generalizations of many of the graph moves listed above. In Section 6, we provide a move-by-move analysis of the spectra of digraphs under these moves. Specifically, we indicate which of the moves defined in [16] for k-graphs preserve the spectra under consideration.

Finally, we observe that a question that is beyond the scope of this paper and that remains open for further investigation is: does there exist as sequence of graph moves that characterizes digraph isospectrality? For instance, some such moves have been identified for the Hermitian spectrum in [26, 38].

This paper is organized as follows. In Section 2, we set the graph terminology that we will use throughout the paper and catalog the various matrices and spectra whose relation to Morita equivalence we will explore. Sections 3 and 4 contain the positive results of our paper. The results of Section 3 allow us to reduce the questions of adjacency isospectrality and line adjacency isospectrality to an enumeration of cycles of a given length. In Section 4, we use these results to show which graph moves do preserve these spectra, proving Propositions 4.2, 4.3, 4.4, and 4.5. Section 5 contains counterexamples that support our negative results. In particular, these examples show which of the various spectra are *not* preserved by the various graph moves. Finally, in Section 6, we give a brief discussion of how our result extend to the case of k-graphs.

Many of the examples in Section 5 were produced using *Mathematica* [29]. *Mathematica* notebooks implementing the moves and spectra considered here are available from the authors upon request.

1.1. **Acknowledgements.** C.F. would like to thank the sabbatical program at the University of Colorado at Boulder and was partially supported by the Simons Foundation Collaboration Grant for

Mathematicians #523991. C.S. would like to thank the sabbatical program at Rhodes College and was partially supported by the E.C. Ellett Professorship in Mathematics.

2. Digraphs and Their Spectra

2.1. **Background and notation for digraphs.** We focus in this paper on *finite digraphs*, i.e., directed graphs with finitely many vertices and finitely many edges that may have loops and parallel edges. We clarify the language with the following.

Definition 2.1 (digraph, multidigraph, simple digraph)**.** A *digraph* $D = (V, E, r, s)$ consists of a set of *vertices* V, a set of *edges* E, and functions $s\colon V \to E$ and $r\colon V \to E$ called the *source* and *range*, respectively. In a fixed digraph D, a *loop* is an edge $e \in E$ such that $r(e) = s(e)$, and two edges $e \neq f$ are *parallel* if $s(e) = s(f)$ and $r(e) = r(f)$. A *simple directed graph* or *simple digraph* is a digraph with no loops nor parallel edges, and a *multidigraph* is a digraph that may have parallel edges but contains no loops. A digraph is *finite* if V and E are both finite sets.

Note that, for simplicity, we use the term *digraph* to refer to what is sometimes called a *pseudodigraph*, as some authors require that digraphs have no loops nor multiple edges. We will use *simple digraph* when we would like to emphasize the absence of loops and multiple edges, and *multidigraph* when we would like to emphasize the absence of loops. For simplicity, we will sometimes say that an edge $e \in E$ such that $s(e) = v$ and $r(e) = w$ is an edge *from v to w*. We use the notation $|S|$ to denote the cardinality of the set S.

Following [19], we have the following definitions for a digraph $D = (V, E, r, s)$.

Definition 2.2 (path, cycle, simple cycle, vertex-simple cycle, exit, return path)**.** A *path of length n* is a finite sequence $(e_1, \ldots, e_n)$ of edges with $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots n-1$. A *cycle* is a nonempty path $(e_1, \ldots, e_n)$ such that $r(e_n) = s(e_1)$. A cycle of length n is *simple* if $e_i \neq e_j$ for any $i, j = 1, \ldots, n$ with $i \neq j$ and it is *vertex-simple* if $r(e_i) \neq r(e_j)$ for any $i, j = 1, \ldots, n$ with $i \neq j$. We say that a vertex-simple path $(e_1, \ldots, e_n)$ has an *exit* if there is an edge f such that $s(f) = s(e_i)$ for some $i = 1, \ldots, n$ with $f \neq e_i$. A cycle $(e_1, \ldots, e_n)$ is a *return path* if $r(e_n) \neq r(e_i)$ for any $i = 1, \ldots, n-1$.

We also have the following for vertices.

Definition 2.3 (source, sink, regular vertex)**.** A *source* is a vertex v such that $r^{-1}(v) = \varnothing$. A *sink* is a vertex v such that $s^{-1}(v) = \varnothing$. A *regular vertex* is a vertex for which $s^{-1}(v)$ is finite and nonempty.

We will make use of the following two constructions of digraphs from a given digraph. Note that the line digraph is often called the *dual graph* in the literature on C^*-algebras of digraphs, [1], [37, p. 237], while the term *line digraph* appears in the graph theory literature [5, p. 2182], [31, p. 173].

Definition 2.4 (line digraph, unparalleled digraph)**.** Let $D = (V, E, r, s)$ be a digraph.

(i) The *line digraph* $\mathcal{L}(D)$ associated to D is the digraph with vertex set E and edge set given by the set of composable pairs of edges. That is, the edge set of $\mathcal{L}(D)$ is the set of pairs $(e, f) \in E^2$ such that $r(e) = s(f)$, with $s_{\mathcal{L}(D)}\big((e, f)\big) = e$ and $r_{\mathcal{L}(D)}\big((e, f)\big) = f$.

(ii) Define an equivalence relation $\sim$ on E by saying $e \sim f$ if $s(e) = s(f)$ and $r(e) = r(f)$. The *unparalleled digraph* $\mathcal{UP}(D)$ associated to D is the digraph with vertex set V and edge set $E/\sim$, where the source and range of $\mathcal{UP}(D)$ are those inherited from D.

It follows from the definition that $\mathcal{L}(D)$ equals $\mathcal{UP}(\mathcal{L}(D))$ for any digraph D, i.e., $\mathcal{UP}(D)$ may have loops but has no parallel edges.

We will sometimes restrict our attention to the following class of digraphs.

Definition 2.5 (strongly connected digraph)**.** A digraph $D = (V, E, r, s)$ is *strongly connected* if for each pair of vertices $v, w \in V$, there is a path from v to w.

Finally, we will use the following various notions of degree for elements of V based on those defined in [5], [6, p. 53], and [11]. Note that our notions of binary indegree and binary outdegree correspond to the indegree and outdegree defined in [10].

Definition 2.6 (indegree, outdegree, binary indegree, binary outdegree)**.** Let $D = (V, E, r, s)$ be a finite digraph. If $v \in V$, the *indegree of* v, denoted $\mathrm{d}_v^{\mathrm{in}}$, is $|r^{-1}(v)|$, the number of edges with range v, and the *outdegree of* v $\mathrm{d}_v^{\mathrm{out}}$, is $|s^{-1}(v)|$, the number of edges with source v. The *binary indegree of* v denoted $\mathrm{d}_v^{\mathrm{b,in}}$, is $\left|s\big(r^{-1}(v)\big)\right|$, the number of distinct w such that there is an edge from w to v, and the *binary outdegree of* v $\mathrm{d}_v^{\mathrm{b,out}}$, is $\left|r\big(s^{-1}(v)\big)\right|$, the number of distinct vertices w such that there is an edge from v to w.

If D is simple, it is easy to see that $\mathrm{d}_v^{\mathrm{b,in}} = \mathrm{d}_v^{\mathrm{in}}$ and $\mathrm{d}_v^{\mathrm{b,out}} = \mathrm{d}_v^{\mathrm{out}}$ for each $v \in V$. For a general digraph D, the binary outdegree $\mathrm{d}_v^{\mathrm{b,out}}$ of $v \in V$ is the outdegree of the vertex v in the unparalleled graph $\mathcal{UP}(D)$, and similarly $\mathrm{d}_v^{\mathrm{b,in}}$ in D is equal to $\mathrm{d}_v^{\mathrm{in}}$ with respect to $\mathcal{UP}(D)$.

2.2. **Matrices and spectra associated to digraphs.** Suppose that $D = (V, E, r, s)$ is a finite digraph. There are a variety of matrices and corresponding eigenvalue spectra that have been associated to D. Before stating the formal definitions, let us briefly discuss the appearances of these spectra and explain our terminology. Note that the authors of some of the publications cited below consider only simple digraphs, multidigraphs, etc., but the definitions extend readily to the case of a general finite digraph. In some cases, we consider two such generalizations, one binary and one non-binary, as described below.

The matrices most commonly associated to D are the *Laplacian* or *Kirchhoff matrix*, defined in terms of the *incidence matrix* (see [3, Def. 4.2, Prop 4.8] and [43, Def. 9.5, Lem. 9.6]), and the *adjacency matrix*. Authors differ on whether the adjacency matrix takes parallel edges into consideration ([6, p. 53, Sec. 3.6], [34, p. 1]), in which case the values of the adjacency matrix are nonnegative integers; or ignores parallel edges, ([27], [32], [7, Sec. 1.7]), in which case the entries are elements of $\{0, 1\}$. For our purposes, both cases will be of interest. Specifically, the moves we will consider in Section 4 can add or remove parallel edges; an example is illustrated in Figure 4, where the application of Move (O) splits the parallel loops at vertex v_2 (and hence the inverse of Move (O) can introduce parallel loops). Following the convention established in Definition 2.6, we will use the term *binary* to indicate a matrix, spectrum, degree, etc. that ignores parallel edges and hence depends only on the unparalleled digraph $\mathcal{UP}(D)$. Hence, we consider both the *adjacency matrix* and the *binary adjacency matrix*. We also consider the *symmetric adjacency spectrum*, that of the product of the adjacency matrix and its transpose, which was studied in [31] and appeared in [5] as the singular value decomposition of the adjacency matrix. This can be defined in a binary and non-binary sense as well. The adjacency matrix of the line digraph $\mathcal{L}(D)$, here called the *line adjacency matrix*, has appeared for instance in [5, p.2182], and we will see that its spectrum is closely related to that of the adjacency matrix. One could also consider a binary line adjacency matrix as the adjacency matrix of the line digraph associated to the unparalleled digraph $\mathcal{UP}(D)$, but we consider this unmotivated and redundant, because a consequence of [5, p. 2183] or Proposition 3.4 below is that this spectrum coincides with the spectrum of the binary adjacency matrix up to the addition of zeros. As the line digraph has no multiple edges, its adjacency matrix is equal to its binary adjacency matrix, so the other interpretation of a "binary line adjacency matrix" is as well redundant.

More recently, the *Hermitian adjacency matrix* and its spectrum were introduced in [26, 35] and studied further in [39, 45]. The graphs considered in [26, 39] are *mixed graphs*, which have both directed and undirected edges. However, as noted in [45, Sec. 2.1], each undirected edge can be replaced with two directed edges, one in each direction, yielding a digraph with the same Hermitian adjacency matrix. The *skew adjacency matrix* [8, 25] and related *skew Laplacian* [25] are defined for digraphs formed by orienting the edges of a simple (unoriented) graph, and hence are binary in our terminology and ignore pairs of directed edges in opposite directions. However, they admit a non-binary generalization to arbitrary finite digraphs, and we consider both cases.

A final recent addition to the literature is the *normalized Laplacian* and related *combinatorial Laplacian* [10, 11] and [7, Sec. 5.4.1]. These matrices are defined for strongly connected digraphs, and their definition relies on this hypothesis in an essential way. Chung and Butler consider *weighted* digraphs, but here we consider the two cases that are intrinsic to D, the case where each edge has weight 1, and the binary case where, from every set of parallel edges, one edge has weight 1 and the others have weight 0.

We collect the formal definitions of these matrices in the following.

Definition 2.7 (matrices associated to a digraph)**.** Let $D = (V, E, r, s)$ be a finite digraph, and fix a linear order of V.

(i) The *incidence matrix* $\mathbf{M}(D) = (m_{ve})_{v\in V, e\in E}$ of D is the matrix whose rows are indexed by V, columns are indexed by E, and whose entries are given by

$$m_{ve} = \begin{cases} +1, & s(e) = v \neq r(e), \\ -1, & r(e) = v \neq s(e), \\ 0 & \text{otherwise.} \end{cases}$$

The *Laplacian* $\Delta(D)$, also called the *Kirchhoff matrix*, is the matrix defined by $\Delta(D) = \mathbf{M}(D)\mathbf{M}(D)^{\mathrm{T}}$. Some authors define $\mathbf{M}(D)$ to be the negative of that given here, but clearly $\Delta(D)$ is independent of this choice.

(ii) The *adjacency matrix* $\mathbf{A}(D) = (a_{vw})_{v,w\in V}$ of D is the square matrix whose rows and columns are indexed by V such that $a_{vw} = |\big(s^{-1}(v) \cap r^{-1}(w)\big)|$, the number of edges $e \in E$ with $s(e) = v$ and $r(e) = w$.

The *binary adjacency matrix* $\mathbf{A}_{\mathrm{b}}(D) = (a_{\mathrm{b},vw})_{v,w\in V}$ of D is given by $\mathbf{A}_{\mathrm{b}}(D) = \mathbf{A}\big(\mathcal{UP}(D)\big)$. That is, $\mathbf{A}_{\mathrm{b}}(D)$ is the square matrix whose rows and columns are indexed by V such that $a_{vw} = +1$ if there is an edge $e \in E$ with $s(e) = v$ and $r(e) = w$, and 0 otherwise.

(iii) The *line adjacency matrix* $\mathbf{L}(D) = (b_{ef})_{e,f\in E}$ of D is given by $\mathbf{L}(D) = \mathbf{A}\big(\mathcal{L}(D)\big)$. That is, $\mathbf{L}(D)$ is the square matrix whose rows and columns are indexed by E such that $b_{ef} = +1$ if $r(e) = s(f)$ and 0 otherwise.

(iv) The *Hermitian adjacency matrix* $\mathbf{H}(D) = (h_{vw})_{v,w\in V}$ of D is the square matrix whose rows and columns are indexed by V such that

$$a_{vw} = \begin{cases} +1, & s^{-1}(v) \cap r^{-1}(w) \neq \varnothing \text{ and } s^{-1}(w) \cap r^{-1}(v) \neq \varnothing, \\ +i, & s^{-1}(v) \cap r^{-1}(w) \neq \varnothing \text{ and } s^{-1}(w) \cap r^{-1}(v) = \varnothing, \\ -i, & s^{-1}(v) \cap r^{-1}(w) = \varnothing \text{ and } s^{-1}(w) \cap r^{-1}(v) \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

(v) The *skew adjacency matrix* $\mathbf{S}(D) = (s_{vw})_{v,w\in V}$ of D is the square matrix whose rows and columns are indexed by V such that $s_{vw} = \big|s^{-1}(v) \cap r^{-1}(w)\big| - \big|s^{-1}(w) \cap r^{-1}(v)\big|$. That is, $\mathbf{S}(D) = \mathbf{A}(D) - \mathbf{A}(D)^{\mathrm{T}}$.

The *binary skew adjacency matrix* $\mathbf{S}_{\mathrm{b}}(D) = (s_{\mathrm{b},vw})_{v,w\in V}$ of D is given by $\mathbf{S}_{\mathrm{b}}(D) = \mathbf{S}\big(\mathcal{UP}(D)\big)$. That is, $\mathbf{S}_{\mathrm{b}}(D)$ is the square matrix whose rows and columns are indexed by V such that

$$s_{\mathrm{b},vw} = \begin{cases} +1, & s^{-1}(v) \cap r^{-1}(w) \neq \varnothing \text{ and } s^{-1}(w) \cap r^{-1}(v) = \varnothing, \\ -1, & s^{-1}(v) \cap r^{-1}(w) = \varnothing \text{ and } s^{-1}(w) \cap r^{-1}(v) \neq \varnothing, \\ 0, & \text{otherwise,} \end{cases}$$

and hence $\mathbf{S}_{\mathrm{b}}(D) = \mathbf{A}_{\mathrm{b}}(D) - \mathbf{A}_{\mathrm{b}}(D)^{\mathrm{T}}$.

(vi) The *skew Laplacian matrix* $\Delta_S(D)$ is given by $\mathrm{diag}(\mathrm{d}_v^{\mathrm{out}} - \mathrm{d}_v^{\mathrm{in}}) - \mathbf{S}(D)$ where $\mathrm{diag}(\mathrm{d}_v^{\mathrm{out}} - \mathrm{d}_v^{\mathrm{in}})$ is the diagonal matrix with rows and columns indexed by V whose vv-entry is the difference between the outdegree and indegree of v.

The *binary skew Laplacian matrix* $\Delta_{\mathrm{b},S}(D)$ is given by $\mathrm{diag}(\mathrm{d}_v^{\mathrm{b,out}} - \mathrm{d}_v^{\mathrm{b,in}}) - \mathbf{S}_{\mathrm{b}}(D)$ where $\mathrm{diag}(\mathrm{d}_v^{\mathrm{b,out}} - \mathrm{d}_v^{\mathrm{b,in}})$ is the diagonal matrix with rows and columns indexed by V whose vv-entry is the difference between the binary outdegree and binary indegree of v.

(vii) The *transition probability matrix* $\mathbf{P}(D) = (p_{vw})_{v,w\in V}$ of D is the square matrix whose rows and columns are indexed by V and such that

$$p_{vw} = \begin{cases} \frac{|s^{-1}(v)\cap r^{-1}(w)|}{\mathrm{d}_v^{\mathrm{out}}}, & \text{there is an edge from } v \text{ to } w,\\ 0, & \text{otherwise.}\end{cases}$$

Note that p_{vw} is the probability of moving from v to w if each edge is equally likely.
Assume D is strongly connected, which is equivalent to the transition probability matrix $\mathbf{P}(D)$ being irreducible. The *Perron-Frobenius vector* $\phi = \phi(D)$ is the unique left-eigenvector of $\mathbf{P}(D)$ with positive entries that sum to 1. Let $\Phi = \Phi(D)$ be the diagonal matrix with entries given by those of ϕ. The *normalized Laplacian* $\Delta_N(D)$ is given by

$$\Delta_N(D) = I_{|V|} - \frac{\Phi^{1/2}\mathbf{P}(D)\Phi^{-1/2} + \Phi^{-1/2}\mathbf{P}(D)^{\mathrm{T}}\Phi^{1/2}}{2},$$

where $I_{|V|}$ is the $|V| \times |V|$ identity matrix, and the *combinatorial Laplacian* $\Delta_C(D)$ is given by

$$\Delta_C(D) = \Phi - \frac{\Phi\mathbf{P}(D) + \mathbf{P}(D)^{\mathrm{T}}\Phi}{2}.$$

The *binary transition probability matrix* $\mathbf{P}_{\mathrm{b}}(D) = (p_{\mathrm{b},vw})$ is the square matrix whose rows and columns are indexed by V and such that

$$p_{\mathrm{b},vw} = \begin{cases} \frac{1}{\mathrm{d}_v^{\mathrm{b,out}}}, & \text{there is an edge from } v \text{ to } w,\\ 0, & \text{otherwise.}\end{cases}$$

Hence, $p_{\mathrm{b},vw}$ is the probability of moving from v to w in $\mathcal{UP}(D)$ if every vertex in $r\big(s^{-1}(v)\big)$ is equally likely. In other words, parallel edges do not affect the likelihood of choosing a vertex.
The definitions of the *binary normalized Laplacian* $\Delta_{\mathrm{b},N}(D)$ and *binary combinatorial Laplacian* $\Delta_{\mathrm{b},C}(D)$ are identical to those of the normalized Laplacian $\Delta_N(D)$ and combinatorial Laplacian $\Delta_C(D)$, respectively, but with $\mathbf{P}(D)$ replaced by $\mathbf{P}_{\mathrm{b}}(D)$.

We will omit D from the notation when it is clear from the context, e.g., $\mathbf{M} = \mathbf{M}(D)$, $\Delta = \Delta(D)$, etc.

The spectra we consider are defined in terms of these matrices as follows.

Definition 2.8 (spectra of a digraph)**.** Let $D = (V, E, r, s)$ be a finite digraph with a fixed linear order of V.

(i) The *Laplace spectrum* $\mathrm{Spec}_{\Delta}(D)$ of D is the multiset of eigenvalues of $\Delta(D)$.

(ii) The *adjacency spectrum* $\mathrm{Spec}_{\mathbf{A}}(D)$ of D is the multiset of eigenvalues of $\mathbf{A}(D)$.
The *symmetric adjacency spectrum* $\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D)$ of D is the multiset of eigenvalues of $\mathbf{A}(D)\mathbf{A}(D)^{\mathrm{T}}$.
The *binary adjacency spectrum* $\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D)$ of D is the multiset of eigenvalues of $\mathbf{A}_{\mathrm{b}}(D)$.
The *symmetric binary adjacency spectrum* $\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D)$ of D is the multiset of eigenvalues of $\mathbf{A}_{\mathrm{b}}(D)\mathbf{A}_{\mathrm{b}}(D)^{\mathrm{T}}$.

(iii) The *line adjacency spectrum* $\mathrm{Spec}_{\mathbf{L}}(D)$ of D is the multiset of eigenvalues of $\mathbf{L}(D)$.

(iv) The *Hermitian adjacency spectrum* $\mathrm{Spec}_{\mathbf{H}}(D)$ of D is the multiset of eigenvalues of $\mathbf{H}(D)$.

(v) The *skew adjacency spectrum* $\mathrm{Spec}_{\mathbf{S}}(D)$ of D is the multiset of eigenvalues of $\mathbf{S}(D)$.
The *binary skew adjacency spectrum* $\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D)$ of D is the multiset of eigenvalues of $\mathbf{S}_{\mathrm{b}}(D)$.

(vi) The *skew Laplace spectrum* $\mathrm{Spec}_{\Delta_S}(D)$ is the multiset of eigenvalues of $\Delta_S(D)$.
The *binary skew Laplace spectrum* $\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D)$ is the multiset of eigenvalues of $\Delta_{\mathrm{b},S}(D)$.

Suppose further that D is strongly connected.

(vii) The *normalized Laplace spectrum* $\mathrm{Spec}_{\Delta_N}(D)$ of D is the multiset of eigenvalues of $\Delta_N(D)$.
The *binary normalized Laplace spectrum* $\mathrm{Spec}_{\Delta_{\mathrm{b},N}}(D)$ of D is the multiset of eigenvalues of $\Delta_{\mathrm{b},N}(D)$.
The *combinatorial Laplace spectrum* $\mathrm{Spec}_{\Delta_C}(D)$ of D is the multiset of eigenvalues of $\Delta_C(D)$.
The *binary combinatorial Laplace spectrum* $\mathrm{Spec}_{\Delta_{\mathrm{b},C}}(D)$ of D is the multiset of eigenvalues of $\Delta_{\mathrm{b},C}(D)$.

Note that in Definition 2.8(ii), we could also consider the "right symmetric adjacency spectrum" defined as the spectrum of $\mathbf{A}(D)^{\mathrm{T}}\mathbf{A}(D)$. However, the nonzero eigenvalues of $\mathbf{A}(D)\mathbf{A}(D)^{\mathrm{T}}$ and $\mathbf{A}(D)^{\mathrm{T}}\mathbf{A}(D)$ coincide, so this spectrum differs from $\mathrm{Spec}_{\mathbf{A}}^{\mathcal{S}}(D)$ only at the multiplicity of zero; see [31, p. 169]. The same statement holds for the binary counterpart.

3. The adjacency and line adjacency spectra via counting cycles

Let $D = (V, E, r, s)$ be a digraph and let $\mathrm{N}_m(D)$ denote the total number of cycles in D of length m. We have the following, which was proven by Bowen and Lanford [4, Theorem 1]; see also [44, Lemma 6]. The proof of Bowen and Lanford applies without change to the case of digraphs. In particular, though [44] restricts to the case of a strongly connected digraph that is not a cycle, the proof applies to an arbitrary finite digraph.

Proposition 3.1. *Let $D = (V, E, r, s)$ be a finite digraph. Let M be the maximum modulus of elements of $\mathrm{Spec}_{\mathbf{A}}(D)$. Then $\sum_{m=1}^{\infty} \frac{t^m}{m} \mathrm{N}_m(D)$ converges absolutely for $|t| < 1/M$, and*

$$\exp \sum_{m=1}^{\infty} \frac{t^m}{m} \mathrm{N}_m(D) = \det\big(I - t\mathbf{A}(D)\big)^{-1}. \tag{3.1}$$

Proof. For $v, w \in V$, let $\mathrm{N}(m, v, w)$ denote the number of walks from v to w of length m. Note that $\mathrm{N}(1, v, w) = \mathbf{A}(D)_{vw}$. Similarly,

$$\begin{aligned}\mathrm{N}(2, v, w) &= \sum_{u \in V} \mathrm{N}(1, v, u)\,\mathrm{N}(1, u, w)\\ &= \big(\mathbf{A}(D)^2\big)_{vw},\end{aligned}$$

from which it follows that $\mathrm{N}(m, v, w) = \big(\mathbf{A}(D)^m\big)_{vw}$ for each $m > 0$.

The total number of cycles of length m starting at a specific $v \in V$ is $\mathrm{N}(m, v, v) = \big(\mathbf{A}(D)^m\big)_{vv}$ so that

$$\mathrm{N}_m(D) = \sum_{v \in V} \mathrm{N}(m, v, v) = \sum_{v \in V} \big(\mathbf{A}(D)^m\big)_{vv} = \mathrm{Trace}\big(\mathbf{A}(D)^m\big) = \sum_{i=1}^{|V|} \lambda_i^m, \tag{3.2}$$

where the λ_i are the eigenvalues of $\mathbf{A}(D)$, i.e. the elements of $\mathrm{Spec}_{\mathbf{A}}(D)$.

Now consider the Taylor series at $x = 1$

$$-\log x = \sum_{m=1}^{\infty} \frac{(1-x)^m}{m}. \tag{3.3}$$

Exponentiating yields the infinite product representation

$$\frac{1}{x} = \exp \sum_{m=1}^{\infty} \frac{(1-x)^m}{m},$$

and substituting $x = 1 - t\lambda$ yields

$$\frac{1}{1 - t\lambda} = \exp \sum_{m=1}^{\infty} \frac{t^m \lambda^m}{m}.$$

Then we have

$$\begin{aligned}\frac{1}{\prod_{i=1}^{|V|}(1-t\lambda_i)} &= \prod_{i=1}^{|V|}\exp\sum_{m=1}^{\infty}\frac{\lambda_i^m}{m}t^m\\ &= \exp\sum_{m=1}^{\infty}\frac{t^m}{m}\sum_{i=1}^{|V|}\lambda_i^m\\ &= \exp\sum_{m=1}^{\infty}\frac{t^m}{m}\,\mathrm{N}_m(D).\end{aligned}$$

Noting that the series in Equation (3.3) has a radius of convergence of 1 so that $\sum_{m=1}^{\infty}\frac{t^m}{m}\,\mathrm{N}_m(D)$ converges absolutely to $\log 1/\prod_{i=1}^{|V|}(1-t\lambda_i)$ whenever each $|t\lambda_i|<1$ completes the proof. □

As a consequence of Proposition 3.1, the nonzero elements of the adjacency spectrum of a finite digraph D are determined by the $N_m(D)$, i.e., we have the following.

Corollary 3.2. *Let $D_1=(V_1,E_1,r_1,s_1)$ and $D_2=(V_2,E_2,r_2,s_2)$ be two finite digraphs. Suppose that $N_m(D_1)=N_m(D_2)$ for all m. Then the multiset of nonzero elements of $\operatorname{Spec}_{\mathbf{A}}(D_1)$ is equal to the multiset of nonzero elements of $\operatorname{Spec}_{\mathbf{A}}(D_2)$.*

Proof. The right-hand side $\det\big(I-t\mathbf{A}(D)\big)^{-1}$ of Equation (3.1) is a rational function in t and hence meromorphic on $\mathbb{C}$; its poles are the nonzero elements of $\operatorname{Spec}_{\mathbf{A}}(D)$ with pole order corresponding to multiplicity. Then as the left-hand side $\exp\sum_{m=1}^{\infty}t^m\,\mathrm{N}_m(D)/m$ converges on a non-empty open set, by [41, Corollary, p. 209], the right-hand side is the unique analytic continuation to its domain of the left-hand side. In particular, the right-hand side is determined by the left-hand side, implying that the set of $\mathrm{N}_m(D)$ determines the nonzero elements of $\operatorname{Spec}_{\mathbf{A}}(D)$. □

In fact, we can strengthen Corollary 3.2 using elementary symmetric polynomials with the following.

Corollary 3.3. *Let $D_1=(V_1,E_1,r_1,s_1)$ and $D_2=(V_2,E_2,r_2,s_2)$ be two finite digraphs. Suppose that $N_m(D_1)=N_m(D_2)$ for all $m\leqslant\min(|V_1|,|V_2|)$. Then the multiset of nonzero elements of $\operatorname{Spec}_{\mathbf{A}}(D_1)$ is equal to the multiset of nonzero elements of $\operatorname{Spec}_{\mathbf{A}}(D_2)$.*

Proof. Note that by Equation (3.2), for $j=1,2$, the mth power sum $\sum_{i=1}^{|V_j|}\lambda_{j,i}^m$ of the eigenvalues of $\mathbf{A}(D_j)$ is equal to the number $\mathrm{N}_m(D_j)$ of cycles of length m in D_j. The first $|V_j|$ power sums generate the ring of symmetric polynomials in $|V_j|$ variables with coefficients in $\mathbb{C}$ (or any field of characteristic 0), see [36, Ch. I, (2.12)]. As the symmetric polynomials are the invariants under the action of the symmetric group $\mathcal{S}_{|V_j|}$ on $\mathbb{C}^{|V_j|}$ by permuting coordinates, and $\mathcal{S}_{|V_j|}$ is finite so that the orbifold $\mathbb{C}^{|V_j|}/\mathcal{S}_{|V_j|}$ is a good quotient for this action, the symmetric polynomials separate points, see [15, Corollary 2.3.8]. It follows that the values of the first $|V_j|$ power sums determine the $\mathcal{S}_{|V_j|}$-orbit, which is exactly the multiset of eigenvalues $\{\lambda_{j,1},\dots,\lambda_{j,|V_j|}\}=\operatorname{Spec}_{\mathbf{A}}(D_j)$. If $|V_1|=|V_2|$, it follows that $\operatorname{Spec}_{\mathbf{A}}(D_1)=\operatorname{Spec}_{\mathbf{A}}(D_2)$. If $|V_1|<|V_2|$, then $\operatorname{Spec}_{\mathbf{A}}(D_2)$ is given by $\operatorname{Spec}_{\mathbf{A}}(D_1)$ with $|V_2|-|V_1|$ additional zero elements. □

We can now apply Proposition 3.1 to show that the non-zero adjacency spectrum of a digraph $D=(V,E,r,s)$ is equal to the non-zero adjacency spectrum of the line digraph $\mathcal{L}(D)$ of D. Because the adjacency matrices of D and $\mathcal{L}(D)$ are generally not the same size, the spectra are not exactly equal in general. After removing zeros, however, the spectra do match. Alternate proofs of this result can be found in [5, p. 2183], [34, Theorem 1.4.4], and the references contained therein.

Proposition 3.4 ([5, 34])**.** *Let $D=(V,E,r,s)$ be a finite digraph. The multiset of non-zero elements of $\operatorname{Spec}_{\mathbf{A}}(D)$ is equal to the multiset of non-zero elements of $\operatorname{Spec}_{\mathbf{A}}(\mathcal{L}(D))$.*

Proof. By Corollary 3.2 or 3.3, it suffices to show that $N_m(D) = N_m(\mathcal{L}(D))$ for all $m \in \mathbb{N}$.

Suppose that $C = (e_1, e_2, \dots, e_m)$ is a cycle in D. Thus $r(e_i) = s(e_{i+1})$ for all $i = 1, \cdots m-1$ and $r(e_m) = s(e_1)$. Note that it is possible that $e_i = e_j$ for some $i \neq j$. By the definition of the line digraph, C corresponds to a sequence C' of edges $\big((e_1, e_2), (e_2, e_3), \dots, (e_{m-1}, e_m), (e_m, e_1)\big)$ in $\mathcal{L}(D)$. Furthermore, because C is a cycle of length m of composable edges in D, it follows that C' forms a cycle of length m in $\mathcal{L}(D)$. We show that the correspondence $C \mapsto C'$ is bijection.

By definition of $\mathcal{L}(D)$, each edge in E_D maps to a distinct vertex in $V_{\mathcal{L}(D)}$. Thus, if C_1 and C_2 are distinct cycles in D, the corresponding sequences C_1' and C_2' are distinct cycles in $\mathcal{L}(D)$, so the correspondence is one-to-one. On the other hand, suppose that $C' = \big((e_1, e_2), (e_2, e_3), \dots, (e_{m-1}, e_m), (e_m, e_1)\big)$ is a cycle of length m in $\mathcal{L}(D)$. Then C' by definition of $\mathcal{L}(D)$, $C = (e_1, \dots, e_m)$ is a cycle in D. Thus the correspondence is onto, and we conclude that the number of cycles of length m in D is equal to the number of cycles of length m in $\mathcal{L}(D)$, as desired. □

4. Morita equivalence moves and spectrum

For a digraph $D = (V, E, r, s)$, we consider transformations of the graph, referred to as Moves (S), (R), (I), (O), (C), and (P), following [2, 19, 21, 42]. It was shown in [42] that if pseudeodigraphs D_1 and D_2, each having a finite number of vertices, differ by a sequence of Moves (S), (R), (O), and (I), then the associated C^*-algebras are stably equivalent, and thus Morita equivalent. In [19], this list of moves was extended when it was shown that if D_1 and D_2 differ by Move (C), then they are stably equivalent. Finally, in [21], by introducing Move (P) the authors achieved a full classification: digraphs D_1 and D_2 having finitely many vertices have Morita equivalent C^*-algebras if and only if they differ by a finite sequence of Moves (S), (R), (O), (I), (C), and (P) and their inverses, see [21, Theorem 3.1].

For the ease of reading, we describe the moves here. Following this, we apply Proposition 3.1 to determine which of the moves preserve the non-zero portion of the adjacency spectrum. In Section 5, we give a series of counterexamples that show which of the spectra of Definition 2.8 are *not* preserved by the various moves.

4.1. **Move (S), remove a regular source.** (See [42, Section 3].) Let $D = (V, E, r, s)$ be a digraph, and let $v \in V$ be a regular vertex that is a source. Define a psuedodigraph $D^{(S)} = (V^{(S)}, E^{(S)}, r^{(S)}, s^{(S)})$ by

$$V^{(S)} := V \backslash \{v\},\ E^{(S)} := E \backslash \{s^{-1}(v)\},\ r^{(S)} := r|_{V_S},\ s^{(S)} := s|_{V_S}.$$

4.2. **Move (R), reduce at a regular vertex.** (See [42, Section 3].) Let $D = (V, E, r, s)$ be a digraph, and let $v \in V$ be a regular vertex that $s^{-1}(v)$ and $s(r^{-1}(v))$ are one-point sets. Let u be the only vertex that emits to v and let f be the only edge v emits. Define a digraph $D^{(R)} = (V^{(R)}, E^{(R)}, r^{(R)}, s^{(R)})$ by

$$V^{(R)} := V \backslash \{v\} \text{ and } E^{(R)} := \Big(E \backslash \Big(r^{-1}(v) \bigcup \{f\}\Big)\Big) \bigcup \left\{ef \,|\, e \in r^{-1}(v)\right\},$$

with range and source maps that extend those of D and satisfy $r^{(R)}(ef) = r(f)$ and $s^{(R)}(ef) = s(e) = u$.

4.3. **Move (O), outsplit at a non-sink.** (See [2, Section 3] and [42, Section 3].) Let $D = (V, E, r, s)$ be a digraph, and let $v \in V$. Suppose that v is not a sink. Partition $s^{-1}(v)$ into a finite number, n say, of sets $\{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n\}$. Define a digraph $D^{(O)} = (V^{(O)}, E^{(O)}, r^{(O)}, s^{(O)})$ by:

$$\begin{aligned} V^{(O)} &= (V \backslash \{v\}) \cup \{v^1, v^2, \dots, v^n\} \text{ and} \\ E^{(O)} &= \big(E \backslash r^{-1}(v)\big) \cup \{e^1, e^2, \dots, e^n \mid e \in E, r(e) = v\}. \end{aligned}$$

For $e \notin r^{-1}(v)$ let $r^{(O)}(e) = r(e)$ and for $e \in r^{-1}(v)$ let $r^{(O)}(e^i) = v^i$ for $i = 1, 2, \dots, n$. For $e \notin s^{-1}(v)$ let $s^{(O)}(v) = s(e)$, for $e \in s^{-1}(v) \backslash r^{-1}(v)$ let $s^{(O)}(e) = v^i$ if $e \in \mathcal{E}_i$, and for $e \in s^{-1}(v) \cap r^{-1}(v)$ let $s^{(O)}(e^j) = v^i$ if $e \in \mathcal{E}_i$, for $i, j = 1, 2, \dots, n$.

4.4. **Move (I), insplit at a regular non-source.** (See [2, Section 5] and [42, Section 3].) Let $D = (V, E, r, s)$ be a digraph, and let $v \in V$. Suppose that v is not a source. Partition $r^{-1}(v)$ into a finite number, n say, of sets $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\}$.6[(Define a digraph $D^{(I)} = (V^{(I)}, E^{(I)}, r^{(I)}, s^{(I)})$ by:

$$V^{(I)} = (V \backslash \{v\}) \cup \{v^1, v^2, \ldots, v^n\} \text{ and}$$
$$E^{(I)} = \big(E \backslash s^{-1}(v)\big) \cup \{e^1, e^2, \ldots, e^n \mid e \in E, s(e) = v\}.$$

For $e \notin r^{-1}(v)$ let $r^{(I)}(e) = r(e)$, for $e \in r^{-1}(v) \backslash s^{-1}(v)$ let $r^{(I)}(e) = v^i$ for $e \in \mathcal{E}_i$, and for $e \in r^{-1}(v) \cap s^{-1}(v)$ let $r^{(I)}(e^j) = v^i$ if $e \in \mathcal{E}_i$ for $i, j = 1, 2, \ldots, n$. For $e \notin s^{-1}(v)$ let $s^{(I)}(v) = s(e)$ and for $e \in s^{-1}(v)$ let $s^{(I)}(e^i) = v^i$ for $i = 1, 2, \ldots, n$.

4.5. **Move (C), the Cuntz splice at a vertex admitting at least two distinct return paths.** (See [19, Section 2] and [42, Section 3].) Let $D = (V, E, r, s)$ be a digraph, and let $v \in V$ be a regular vertex that supports at least two return paths. Define a digraph $D^{(C)} = (V^{(C,v)}, E^{(C,v)}, r^{(C,v)}, s^{(C,v)})$ by

$$V^{(C,v)} = V \cup \{u_1, u_2\} \text{ and}$$
$$E^{(C,v)} = E \cup \{e_1, e_2, f_1, f_2, h_1, h_2\},$$

where $r^{(C,v)}|_E = r$, $s^{(C,v)}|_E = s$,

$$r^{(C,v)}(e_1) = u_1, \quad r^{(C,v)}(e_2) = v, \quad r^{(C,v)}(f_i) = u_i, \quad r^{(C,v)}(h_i) = u_i,$$

and

$$s^{(C,v)}(e_1) = v, \quad s^{(C,v)}(e_2) = u_1, \quad s^{(C,v)}(f_i) = u_1, \quad s^{(C,v)}(h_i) = u_2.$$

We say that $D^{(C,v)}$ is the digraph obtained from D by performing Move (C) at vertex v.

If S is a subset of E such that each $w \in S$ is a regular vertex supporting at least two return paths, then we may perform Move (C) at each $w \in S$. We label the resulting digraph $D^{(C,S)}$, and in this case refer to the additional vertices by u_i^w, $i = 1, 2$, for each $w \in S$.

Remark 4.1. We note that the Cuntz move can in theory be performed at any vertex, regardless of whether it has at least two distinct return paths. The condition on return paths is required in order to conclude that the corresponding C^*-algebras are Morita equivalent, see [21, Section 2.5].

4.6. **Move (P), eclose a cyclic component.** (See [21, Section 2].) Let $D = (V, E, r, s)$ be a digraph and let $v \in V$ support a loop but no other return path. Suppose that the loop has an exit. Note that these conditions imply that there is at least one vertex $w \in V$ distinct from v such that there is an edge e with $s(e) = v$ and $r(e) = w$.

Let $S = \{w \in V \backslash \{v\} \,|\, s^{-1}(v) \cap r^{-1}(w) \neq \varnothing\}$. Suppose further that if $w \in S$, then w is a regular vertex that supports at least two return paths. Define a digraph $D^{(P,v)} = (V^{(P,v)}, E^{(P,v)}, r^{(P,v)}, s^{(P,v)})$ by

$$V^{(P,v)} = V^{(C,S)}$$
$$E^{(P,v)} = E^{(C,S)} \cup \{\bar{e}_w, \tilde{e}_w \,|\, w \in S, e \in s^{-1}(v) \cap r^{-1}(w)\},$$

where $r^{(P,v)}|_{E^{(C,S)}} = r^{(C,S)}$, $s^{(P,v)}|_{E^{(C,S)}} = s^{(C,S)}$, $r^{(P,v)}(\bar{e}_w) = r^{(P,v)}(\tilde{e}_w) = v_2^w$, and $s^{(P,v)}(\bar{e}_w) = s^{(P,v)}(\tilde{e}_w) = v$.

We say that $D^{(P,v)}$ is the digraph formed by performing Move (P) at v.

As in Remark 4.1, Move (P) can in theory be performed at any vertex. The requirement in the definition of Move (P) that vertex w be a regular vertex that supports at least two return paths ensures that when two digraphs differ by Move (P), their corresponding C^*-algebras are Morita equivalent.

4.7. **Moves that preserve spectra.** Based on these descriptions of the moves given above, we can now prove the positive part of Theorem 1.1, which we organize into three propositions. We begin with the following.

Proposition 4.2. *Given a finite digraph D, let $D^{(S)}$, $D^{(O)}$, and $D^{(I)}$ be digraphs resulting from performing Move (S), (O), and (I) to D respectively. Then the multisets of nonzero elements in* $\mathrm{Spec}_{\mathbf{A}}(D)$, $\mathrm{Spec}_{\mathbf{A}}(D^{(S)})$, $\mathrm{Spec}_{\mathbf{A}}(D^{(O)})$, *and* $\mathrm{Spec}_{\mathbf{A}}(D^{(I)})$ *are all equal.*

Proof. The result follows directly from either Corollary 3.2 or 3.3.

For Move (S), since the vertex that is removed is a source, it is not part of any cycles, and therefore the number of cycles of a given length in D is equal to the number of cycles of that length in $D^{(S)}$.

Now, suppose that $D^{(O)}$ is the digraph that results from performing Move (O) to D at vertex v. Then there is a bijection between the cycles of length m in D and the cycles of length m in $D^{(O)}$ as follows. Suppose that $e_1, e_2, \ldots, e_m$ is a collection of edges in D forming a cycle $(e_1, e_2, \ldots, e_m)$. (Note that it may be the case that $e_j = e_k$ for $j \neq k$.). If $s(e_j) \neq v$ for $j = 1, \ldots, m$, then by definition of Move (O), this cycle is mapped to exactly one cycle, namely $(e_1, e_2, \ldots, e_m)$ in $D^{(O)}$. If $s(e_j) = v$ for some $j = 1, \ldots, m$, suppose without loss of generality that $s(e_1) = v$. Using notation from above, suppose further that $e_1 \in \mathcal{E}_i$. Then the cycle $(e_1, e_2, \ldots, e_m)$ in D is mapped to cycle $(e_1, e_2, \ldots, e_m^i)$ in $D^{(O)}$, where $r(e_m^i) = s(e_1) = v^i$. Again, by definition of Move (O), this correspondence is one-to-one.

On the other hand, suppose that $(e_1, e_2, \ldots, e_m)$ is a cycle of length m in $D^{(O)}$. If $s(e_j) \neq v^i$ for any $j = 1, \ldots, m$, $i = 1, \ldots, n$, then, as above $(e_1, e_2, \ldots, e_m)$ in $D^{(O)}$ is the image under Move (O) of $(e_1, e_2, \ldots, e_m)$ in D. If $s(e_j) = v^i$ for some $j = 1, \ldots, m$, $i = 1, \ldots, n$, suppose without loss of generality that $s(e_1) = v^i$ for some $i = 1, \ldots, n$. Then $e_m = e^i$ for some edge e in D with $r(e) = v$, and $(e_1, e_2, \ldots, e_m) = (e_1, \ldots, e_{m-1}, e^i)$ in $D^{(O)}$ is the image under Move (O) of cycle $(e_1, e_2, \ldots, e_{m-1}, e)$ in D. Thus, we have a bijective correspondence between the cycles of length m in D and the cycles of length m in $D^{(O)}$, and by Corollary 3.2 or 3.3, the multisets of nonzero elements in $\mathrm{Spec}_{\mathbf{A}}(D)$ and $\mathrm{Spec}_{\mathbf{A}}(D^{(O)})$ are equal. The argument that the multisets of nonzero elements in $\mathrm{Spec}_{\mathbf{A}}(D)$ and $\mathrm{Spec}_{\mathbf{A}}(D^{(I)})$ are equal is similar. □

Proposition 4.3. *Given a finite digraph D, let $D^{(S)}$, $D^{(O)}$, and $D^{(I)}$ be digraphs resulting from performing Move (S), (O), and (I) to D respectively. Then the multisets of nonzero elements in* $\mathrm{Spec}_{\mathbf{L}}(D)$, $\mathrm{Spec}_{\mathbf{L}}(D^{(S)})$, $\mathrm{Spec}_{\mathbf{L}}(D^{(O)})$, *and* $\mathrm{Spec}_{\mathbf{L}}(D^{(I)})$ *are all equal.*

Proof. The result follows directly from Propositions 3.4 and 4.2. □

Proposition 4.4. *Given a finite digraph D, let $D^{(S)}$ be a digraph resulting from performing Move (S) to D. Then the multisets of nonzero elements in* $\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D)$ *and* $\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D^{(S)})$ *are equal.*

Proof. The binary adjacency spectrum of a graph can be obtained by first replacing all multiple directed edges from vertex v to vertex w with a single directed edge from v to w, then computing the spectrum of the adjacency matrix of the resulting digraph. Because the binary adjacency spectrum is computed using an adjacency matrix, Corollary 3.2 and 3.3 apply; here we need only confirm that the number of cycles of length m in the resulting digraph is equal before and after we perform Move (S). But since Move (S) is the removal of a source, that vertex is not included in any cycles, and thus the number of cycles of length m remains unchanged, as desired. The result follows. □

Proposition 4.5. *Given a finite digraph D, let $D^{(C)}$ be a digraph resulting from performing Move (C) to D. Then the multisets of nonzero elements in* $\mathrm{Spec}_{\mathbf{S}}(D)$ *and* $\mathrm{Spec}_{\mathbf{S}}(D^{(C)})$ *are equal, the multisets of nonzero elements in* $\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D)$ *and* $\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D^{(C)})$ *are equal, the multisets of nonzero elements in* $\mathrm{Spec}_{\Delta_S}(D)$ *and* $\mathrm{Spec}_{\Delta_S}(D^{(C)})$ *are equal, and the multisets of nonzero elements in* $\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D)$ *and* $\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D^{(C)})$ *are equal.*

Proof. Following the notation of Section 4.5, let v be the vertex at which Move (C) is applied and let $V \cup \{u_1, u_2\}$ be the vertex set of $D^{(C)}$. The new vertices and edges that are added by Move (C) are pictured in Figure 1. In particular, note that $\mathrm{d}_v^{\mathrm{out}}$, $\mathrm{d}_v^{\mathrm{in}}$, $\mathrm{d}_v^{\mathrm{b,out}}$, and $\mathrm{d}_v^{\mathrm{b,in}}$ are each increased by 1.

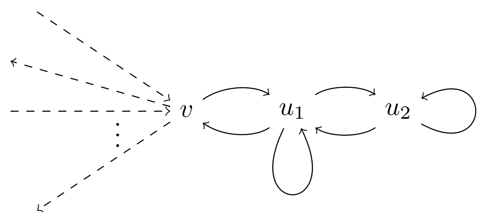


FIGURE 1. The vertices and edges added to a digraph by Move (C).

Furthermore, for each $w, w' \in V$, the w, w'-entries of $\mathbf{S}(D)$ and $\mathbf{S}(D^C)$ coincide, as do the w, w'-entries of $\mathbf{S}_{\mathrm{b}}(D)$ and $\mathbf{S}_{\mathrm{b}}(D^C)$, the diagonal entries $\mathrm{d}_w^{\mathrm{out}} - \mathrm{d}_w^{\mathrm{in}}$ of $\varDelta_S(D)$ and $\varDelta_S(D^{(C)})$, and the diagonal entries $\mathrm{d}_w^{\mathrm{b,out}} - \mathrm{d}_w^{\mathrm{b,in}}$ of $\varDelta_{\mathrm{b},S}(D)$ and $\varDelta_{\mathrm{b},S}(D^{(C)})$. The entries corresponding to u_1 and u_2 are easily seen to vanish so that, ordering the vertex set of $D^{(C)}$ with u_1, u_2 as the last two elements, we have

$$\mathbf{S}(D^{(C)}) = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & \mathbf{S}(D) & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right), \qquad \mathbf{S}_{\mathrm{b}}(D^{(C)}) = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & \mathbf{S}_{\mathrm{b}}(D) & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right),$$

$$\varDelta_S(D^{(C)}) = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & \varDelta_S(D) & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right), \quad \text{and} \quad \varDelta_{\mathrm{b},S}(D^{(C)}) = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & \varDelta_{\mathrm{b},S}(D) & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right)$$

The claim then follows. □

5. NEGATIVE RESULTS

In this section, we give counterexamples to illustrate the failures of the various spectra to be preserved under the remaining moves, completing the proof of Theorem 1.1. Specifically, Examples 5.1, 5.2, 5.3, 5.4, and 5.5 are sufficient to indicate the failures of moves to preserve spectra stated in Theorem 1.1. Note in particular that the digraphs in Examples 5.1 and 5.2 are not strongly connected, while the digraphs in Examples 5.3, 5.4, and 5.5 are strongly connected. To clarify the organization, Table 1 indicates which counterexample applies to each spectrum and move pair, while Table 2 indicates counterexamples within the class of strongly connected digraphs.

Note that if digraphs D_1 and D_2 are related by Move (S), then one must contain a source and therefore is not strongly connected. Similarly, Move (P) only applies to a digraph containing a vertex u that supports a loop and no other return path, and such that the loop at u has an exit. Therefore, the exit begins with an edge from u to another vertex v which cannot begin a return path, implying that there is no path from v to u and that the digraph is not strongly connected. That is, Moves (S) and (P) do not apply within the class of strongly connected digraphs, and the spectra $\mathrm{Spec}_{\varDelta_N}$, $\mathrm{Spec}_{\varDelta_C}$, $\mathrm{Spec}_{\varDelta_{\mathrm{b},N}}$, and $\mathrm{Spec}_{\varDelta_{\mathrm{b},C}}$ are not defined for at least one element of a pair of digraphs connected by Move (S) or (P).

Example 5.1. Let D_1 denote the digraph of [21, Fig. 1(a)] (there denoted E); see Figure 2A. Let $D_1^{(S)}$ denote the digraph obtained from D_1 by applying the inverse of Move (S), adjoining a source v_s that has edges to v_1, v_2, and v_3 (Figure 2B); let $D_1^{(R)}$ denote the digraph obtained by applying the inverse of Move (R) to D_1, adjoining a vertex v_r with edges to and from v_2 (see Figure 2C); let $D_1^{(O)}$ denote be obtained by applying Move (O) at v_1 using the partition with two elements, the first

Spectrum	**Move (S)**	**Move (R)**	**Move (O)**	**Move (I)**	**Move (C)**	**Move (P)**
$\mathrm{Spec}_{\boldsymbol{\Delta}}$	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.2
$\mathrm{Spec}_{\mathbf{A}}$	Prop. 4.2	Ex. 5.1	Prop. 4.2	Prop. 4.2	Ex. 5.1	Ex. 5.2
$\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}$	Prop. 4.4	Ex. 5.1	Ex. 5.3	Ex. 5.1	Ex. 5.1	Ex. 5.2
$\mathrm{Spec}_{\mathbf{A}}^{S}$	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.2
$\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}^{S}$	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.2
$\mathrm{Spec}_{\mathbf{L}}$	Prop. 4.3	Ex. 5.1	Prop. 4.3	Prop. 4.3	Ex. 5.1	Ex. 5.2
$\mathrm{Spec}_{\mathbf{H}}$	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.1	Ex. 5.2
$\mathrm{Spec}_{\mathbf{S}}$	Ex. 5.1	Ex. 5.5	Ex. 5.1	Ex. 5.1	Prop. 4.5	Ex. 5.2
$\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}$	Ex. 5.1	Ex. 5.5	Ex. 5.1	Ex. 5.1	Prop. 4.5	Ex. 5.2
Spec_{Δ_S}	Ex. 5.1	Ex. 5.5	Ex. 5.1	Ex. 5.1	Prop. 4.5	Ex. 5.2
$\mathrm{Spec}_{\Delta_{\mathrm{b},S}}$	Ex. 5.1	Ex. 5.5	Ex. 5.1	Ex. 5.1	Prop. 4.5	Ex. 5.2
Spec_{Δ_N}	NA	Ex. 5.4	Ex. 5.3	Ex. 5.3	Ex. 5.4	NA
Spec_{Δ_C}	NA	Ex. 5.4	Ex. 5.3	Ex. 5.3	Ex. 5.4	NA
$\mathrm{Spec}_{\Delta_{\mathrm{b},N}}$	NA	Ex. 5.4	Ex. 5.3	Ex. 5.3	Ex. 5.4	NA
$\mathrm{Spec}_{\Delta_{\mathrm{b},C}}$	NA	Ex. 5.4	Ex. 5.3	Ex. 5.3	Ex. 5.4	NA

TABLE 1. For each move and spectrum pair, the result demonstrating that the move preserves the nonzero elements of the spectrum or a counterexample indicating that the nonzero spectral elements are not preserved. Moves (S) and (P) either must be applied to or produce non-strongly connected digraphs, for which the spectra Spec_{Δ_N}, Spec_{Δ_C}, $\mathrm{Spec}_{\Delta_{\mathrm{b},N}}$, and $\mathrm{Spec}_{\Delta_{\mathrm{b},C}}$ are not defined.

Spectrum	**Move (R)**	**Move (O)**	**Move (I)**	**Move (C)**
$\mathrm{Spec}_{\boldsymbol{\Delta}}$	Ex. 5.4	Ex. 5.4	Ex. 5.4	Ex. 5.4
$\mathrm{Spec}_{\mathbf{A}}$	Ex. 5.4	Prop. 4.2	Prop. 4.2	Ex. 5.4
$\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}$	Ex. 5.4	Ex. 5.3	Ex. 5.3	Ex. 5.4
$\mathrm{Spec}_{\mathbf{A}}^{S}$	Ex. 5.4	Ex. 5.4	Ex. 5.4	Ex. 5.4
$\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}^{S}$	Ex. 5.4	Ex. 5.4	Ex. 5.4	Ex. 5.4
$\mathrm{Spec}_{\mathbf{L}}$	Ex. 5.4	Prop. 4.3	Prop. 4.3	Ex. 5.4
$\mathrm{Spec}_{\mathbf{H}}$	Ex. 5.4	Ex. 5.4	Ex. 5.4	Ex. 5.4
$\mathrm{Spec}_{\mathbf{S}}$	Ex. 5.5	Ex. 5.4	Ex. 5.4	Prop. 4.3
$\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}$	Ex. 5.5	Ex. 5.4	Ex. 5.4	Prop. 4.5
Spec_{Δ_S}	Ex. 5.5	Ex. 5.4	Ex. 5.4	Prop. 4.5
$\mathrm{Spec}_{\Delta_{\mathrm{b},S}}$	Ex. 5.5	Ex. 5.4	Ex. 5.4	Prop. 4.5
Spec_{Δ_N}	Ex. 5.4	Ex. 5.4	Ex. 5.4	Ex. 5.4
Spec_{Δ_C}	Ex. 5.4	Ex. 5.4	Ex. 5.4	Ex. 5.4
$\mathrm{Spec}_{\Delta_{\mathrm{b},N}}$	Ex. 5.4	Ex. 5.4	Ex. 5.4	Ex. 5.4
$\mathrm{Spec}_{\Delta_{\mathrm{b},C}}$	Ex. 5.4	Ex. 5.4	Ex. 5.4	Ex. 5.4

TABLE 2. For each move and spectrum pair, the result demonstrating that the move preserves the nonzero elements of the spectrum or a counterexample indicating that the nonzero spectral elements are not preserved within the class of strongly connected digraphs. Moves (S) and (P) are omitted, as they do not preserve strongly connectedness.

containing the loop at v_1 and edge from v_1 to v_2, the second containing both edges from v_1 to v_3 (see Figure 2D); let $D_1^{(I)}$ denote the digraph obtained by applying Move (I) at v_2 using the partition with two elements, the first containing one loop at v_2 and the edge from v_1 to v_2, the second containing the other loop at v_2 (Figure 2E); and let $D_1^{(C)}$ denote the digraph obtained from D_1 by applying Move (C) to D_1 at vertex v_2 (Figure 2F). Note that D_1 does not satisfy the hypotheses required to apply Move (P) as discussed in Section 4.7; see Remark 4.1. Specifically, v_3 is the range of edges with source v_1 and v_2, respectively, and v_3 supports only one return path, so Move (P) cannot be applied at v_1 nor v_2; since there are no edges from v_3 to any other vertex, the move cannot be applied at v_3. Hence, Move (P) will be considered in Example 5.2 below.

The Laplace spectra of the digraphs formed from D_1 are follows, with numerical approximations given when expressions by radicals are cumbersome:

$$\begin{aligned}
\mathrm{Spec}_{\Delta}(D_1) &= \left\{\frac{15+\sqrt{33}}{2}, 9, \frac{15-\sqrt{33}}{2}\right\},\\
\mathrm{Spec}_{\Delta}(D_1^{(S)}) &= \{\approx 11.4186, 10, \approx 6.38787, \approx 2.19358\},\\
\mathrm{Spec}_{\Delta}(D_1^{(R)}) &= \{\approx 9.05932, 9, \approx 4.59559, \approx 1.34509\},\\
\mathrm{Spec}_{\Delta}(D_1^{(O)}) &= \{\approx 10.592, \approx 7.49276, 6, \approx 1.91524\},\\
\mathrm{Spec}_{\Delta}(D_1^{(I)}) &= \{10, 8+\sqrt{2}, 8-\sqrt{2}, 4\},\\
\mathrm{Spec}_{\Delta}(D_1^{(C)}) &= \left\{9+\sqrt{17}, 9, \frac{13+\sqrt{17}}{2}, 9-\sqrt{17}, \frac{13-\sqrt{17}}{2}\right\},
\end{aligned}$$

Hence, these examples illustrate that the Laplace spectrum, as well as the submultiset of nonzero elements, is not preserved by any of the moves (S), (R), (O), (I), nor (C).

The adjacency spectra of the digraphs formed from D_1 are:

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{A}}(D_1) &= \{2, 1, 1\},\\
\mathrm{Spec}_{\mathbf{A}}(D_1^{(S)}) = \mathrm{Spec}_{\mathbf{A}}(D_1^{(O)}) = \mathrm{Spec}_{\mathbf{A}}(D_1^{(I)}) &= \{2, 1, 1, 0\},\\
\mathrm{Spec}_{\mathbf{A}}(D_1^{(R)}) &= \left\{\frac{1+\sqrt{5}}{2}, 1, 1, \frac{1-\sqrt{5}}{2}\right\},\\
\mathrm{Spec}_{\mathbf{A}}(D_1^{(C)}) &= \{\approx 2.80194, \approx 1.44504, 1, 1, \approx -0.24698\}.
\end{aligned}$$

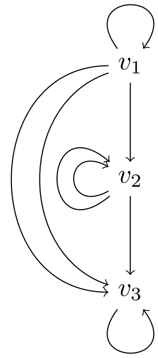
Hence, the submultiset of nonzero elements of the adjacency spectrum is not invariant under moves (R) and (C) and, as expected from Proposition 4.2, is not changed by moves (S), (O), and (I).

The binary adjacency spectra of the digraphs formed from D_1 are:

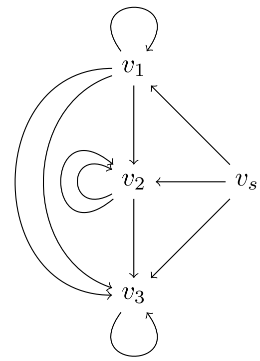
$$\begin{aligned}
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_1) &= \{1, 1, 1\},\\
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_1^{(S)}) = \mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_1^{(O)}) &= \{1, 1, 1, 0\}\\
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_1^{(R)}) &= \left\{\frac{1+\sqrt{5}}{2}, 1, 1, \frac{1-\sqrt{5}}{2}\right\},\\
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_1^{(I)}) &= \{2, 1, 1, 0\},\\
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_1^{(C)}) &= \{1+\sqrt{2}, 1, 1, 1, 1-\sqrt{2}\}.
\end{aligned}$$

The nonzero elements of the binary adjacency spectrum are not preserved by moves (R), (I), and (C); Move (O) will be treated in Example 5.3 below. Move (S) does not change the nonzero elements of $\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_1)$ as indicated by Proposition 4.4.

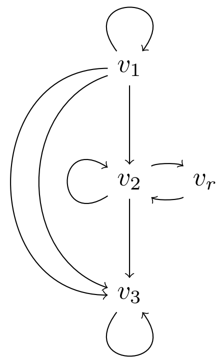
The symmetric adjacency spectra $\mathrm{Spec}_{\mathbf{A}}^{\mathcal{S}}$ and symmetric binary adjacency spectra $\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}^{\mathcal{S}}$ are given below. In each case, the multiset of nonzero elements of the spectrum are not invariant under any of



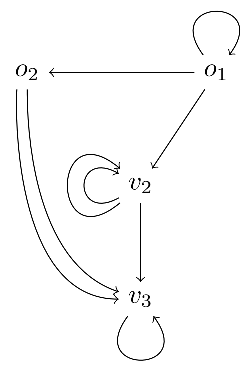
(A) The digraph D_1.



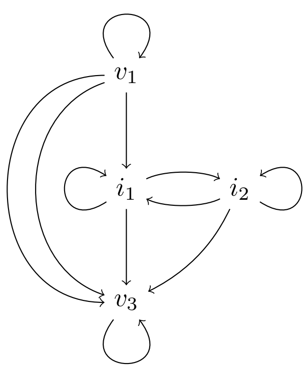
(B) The digraph $D_1^{(S)}$ obtained by applying the inverse of Move (S) to D_1, adjoining source s.



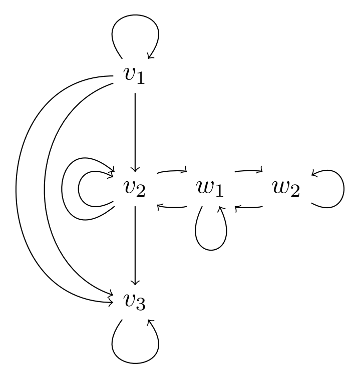
(C) The digraph $D_1^{(R)}$ obtained by applying the inverse of Move (R) to D_1, adjoining the vertex r.



(D) The digraph $D_1^{(O)}$ obtained by applying Move (O) to D_1 at v_1 using the partition whose first element contains the loop at v_1 and edge from v_1 to v_2 and second element contains both edges from v_1 to v_3.



(E) The digraph $D_1^{(I)}$ obtained by applying Move (I) to D_1 at v_2 using the partition whose first element contains one loop at v_2 and the edge from v_1 to v_2 and second contains the other loop at v_2.



(F) The digraph $D_1^{(C)}$ obtained by applying Move (C) to D_1 at vertex v_2.

FIGURE 2. The digraph D_1 and the application of Moves (S), (R), (O), (I), and (C) from Example 5.1.

the moves.

$$\begin{aligned}
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_1) &= \{\approx 10.0494, \approx 1.719, \approx 0.231548\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_1^{(S)}) &= \{\approx 12.7148, \approx 1.74411, \approx 0.541128, 0\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_1^{(R)}) &= \{\approx 8.73968, \approx 1.46182, \approx 0.684079, \approx 0.114421\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_1^{(O)}) &= \{\approx 7.70156, 4, \approx 1.29844, 0\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_1^{(I)}) &= \{\approx 10.8363, \approx 1.86713, \approx 0.296548, 0\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_1^{(C)}) &= \{\approx 11.5864, \approx 4.6524, \approx 1.42213, \approx 0.294867, \approx 0.0442395\};
\end{aligned}$$

$$\begin{aligned}
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_1) &= \{\approx 5.04892, \approx 0.643104, \approx 0.307979\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_1^{(S)}) &= \{\approx 7.89167, \approx 0.785825, \approx 0.322504, 0\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_1^{(R)}) &= \left\{\frac{1+\sqrt{5}}{2}, 1, 1, \frac{1-\sqrt{5}}{2}\right\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_1^{(O)}) &= \left\{4, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, 0\right\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_1^{(I)}) &= \{\approx 8.12071, \approx 1.31922, \approx 0.560067, 0\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_1^{(C)}) &= \{\approx 7.19584, \approx 3.35194, \approx 0.844535, \approx 0.511755, \approx 0.0959274\};
\end{aligned}$$

The line adjacency spectra of the digraphs formed by applying moves to D_1 are:

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{L}}(D_1) &= \{2, 1, 1, 0, 0, 0, 0, 0\},\\
\mathrm{Spec}_{\mathbf{L}}(D_1^{(S)}) = \mathrm{Spec}_{\mathbf{L}}(D_1^{(I)}) &= \{2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0\},\\
\mathrm{Spec}_{\mathbf{L}}(D_1^{(R)}) &= \left\{\frac{1+\sqrt{5}}{2}, 1, 1, \frac{1-\sqrt{5}}{2}, 0, 0, 0, 0, 0\right\},\\
\mathrm{Spec}_{\mathbf{L}}(D_1^{(O)}) &= \{2, 1, 1, 0, 0, 0, 0, 0, 0\},\\
\mathrm{Spec}_{\mathbf{L}}(D_1^{(C)}) &= \{\approx 2.80194, \approx 1.44504, 1, 1, \approx -0.24698, 0, 0, 0, 0, 0, 0, 0, 0, 0\}.
\end{aligned}$$

As guaranteed by Corollaries 3.2 and 3.3, the nonzero elements of $\mathrm{Spec}_{\mathbf{L}}(D_1)$ coincide with the nonzero elements of $\mathrm{Spec}_{\mathbf{A}}(D_1)$. As in the case of $\mathrm{Spec}_{\mathbf{A}}(D_1)$, we see that the multiset of nonzero elements of $\mathrm{Spec}_{\mathbf{L}}(D_1)$ is not invariant under moves (C) and (R) and is invariant under the moves guaranteed by Proposition 4.2.

The Hermitian adjacency spectra are

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{H}}(D_1) &= \{1+\sqrt{3}, 1, 1-\sqrt{3}\},\\
\mathrm{Spec}_{\mathbf{H}}(D_1^{(S)}) &= \{\approx 3.22001, \approx -1.74108, \approx 1.23136, \approx 0.289713\},\\
\mathrm{Spec}_{\mathbf{H}}(D_1^{(R)}) &= \{\approx 2.86081, \approx 1.2541, \approx -1.11491, 0\},\\
\mathrm{Spec}_{\mathbf{H}}(D_1^{(O)}) &= \{\approx 2.81361, \approx -1.34292, 1, \approx 0.529317\},\\
\mathrm{Spec}_{\mathbf{H}}(D_1^{(I)}) &= \{\approx 3.34292, \approx 1.47068, \approx -0.813607, 0\},\\
\mathrm{Spec}_{\mathbf{H}}(D_1^{(C)}) &= \{3, 2, -1, 1, 0\},
\end{aligned}$$

demonstrating that $\mathrm{Spec}_{\mathbf{H}}$ is not preserved by any of these moves.

The skew adjacency spectra, binary skew adjacency spectra, skew Laplace spectra, and binary skew Laplace spectra are

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{S}}(D_1) &= \{i\sqrt{6}, -i\sqrt{6}, 0\},\\
\mathrm{Spec}_{\mathbf{S}}(D_1^{(S)}) &= \{3i, -3i, 0, 0\},\\
\mathrm{Spec}_{\mathbf{S}}(D_1^{(R)}) &= \{i\sqrt{6}, -i\sqrt{6}, 0, 0\},\\
\mathrm{Spec}_{\mathbf{S}}(D_1^{(O)}) = \mathrm{Spec}_{\mathbf{S}}(D_1^{(I)}) &= \left\{i\sqrt{\frac{3\sqrt{5}+7}{2}}, -i\sqrt{\frac{3\sqrt{5}+7}{2}}, i\sqrt{\frac{7-3\sqrt{5}}{2}}, -i\sqrt{\frac{7-3\sqrt{5}}{2}}\right\},\\
\mathrm{Spec}_{\mathbf{S}}(D_1^{(C)}) &= \{i\sqrt{6}, -i\sqrt{6}, 0, 0, 0\};
\end{aligned}$$

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_1) &= \left\{i\sqrt{3}, -i\sqrt{3}, 0\right\},\\
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_1^{(S)}) &= \left\{i\sqrt{2\sqrt{2}+3}, -i\sqrt{2\sqrt{2}+3}, i\sqrt{3-2\sqrt{2}}, -i\sqrt{3-2\sqrt{2}}\right\},\\
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_1^{(R)}) &= \left\{i\sqrt{3}, -i\sqrt{3}, 0, 0\right\},\\
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_1^{(O)}) &= \{2i, -2i, 0, 0\},\\
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_1^{(I)}) &= \left\{i\sqrt{\sqrt{3}+2}, -i\sqrt{\sqrt{3}+2}, i\sqrt{2-\sqrt{3}}, -i\sqrt{2-\sqrt{3}}\right\},\\
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_1^{(C)}) &= \left\{i\sqrt{3}, -i\sqrt{3}, 0, 0, 0\right\};
\end{aligned}$$

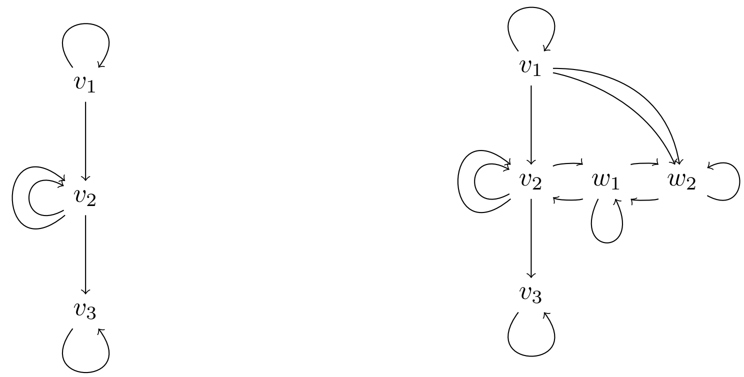
$$\begin{aligned}
\mathrm{Spec}_{\Delta_S}(D_1) &= \left\{-\sqrt{3}, \sqrt{3}, 0\right\},\\
\mathrm{Spec}_{\Delta_S}(D_1^{(S)}) = \mathrm{Spec}_{\Delta_S}(D_1^{(I)}) &= \left\{-\sqrt{3}-1, 2, \sqrt{3}-1, 0\right\},\\
\mathrm{Spec}_{\Delta_S}(D_1^{(R)}) &= \left\{-\sqrt{3}, \sqrt{3}, 0, 0\right\},\\
\mathrm{Spec}_{\Delta_S}(D_1^{(O)}) &= \{0, 0, 0, 0\},\\
\mathrm{Spec}_{\Delta_S}(D_1^{(C)}) &= \left\{-\sqrt{3}, \sqrt{3}, 0, 0, 0\right\};
\end{aligned}$$

$$\begin{aligned}
\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D_1) &= \{-1, 1, 0\},\\
\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D_1^{(S)}) &= \{-2, 2, 0, 0\},\\
\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D_1^{(R)}) &= \{-1, 1, 0, 0\},\\
\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D_1^{(O)}) &= \{0, 0, 0, 0\},\\
\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D_1^{(I)}) &= \{-2, 1, 1, 0\},\\
\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D_1^{(C)}) &= \{-1, 1, 0, 0, 0\}.
\end{aligned}$$

The nonzero elements of these spectra are therefore not preserved by Moves (S), (O), nor (I). That they are preserved by Move (C) follows from Proposition 4.5. The case of Move (R) will be treated in Example 5.5 below.

Note in particular that the digraphs $D_1^{(S)}$, $D_1^{(R)}$, $D_1^{(O)}$, and $D_1^{(I)}$ in Example 5.1 each have four vertices. Hence, as at least one of moves (S), (R), (O), and (I) fail to preserve the nonzero elements of each spectrum under consideration, this example also illustrates the failure of spectra to be preserved when restricting to digraphs with Morita equivalent C^*-algebras and the same number of vertices.

The following demonstrates that none of the spectra are preserved by Move (P).



(A) The digraph D_2.

(B) The digraph $D_2^{(P)}$ obtained by applying Move (P) to D_2 at vertex v_1.

FIGURE 3. The digraph D_2 and the application of Move (P) from Example 5.2.

Example 5.2. Let D_2 denote the digraph in [21, Fig. 1(b)] (there denoted F) pictured in Figure 3A, and let $D_2^{(P)}$ denote the digraph obtained by applying Move (P) to D_2 at v_1. We compute each of the spectra of D_2 and $D_2^{(P)}$ in Definition 2.8 to demonstrate that none of these spectra (or their nonzero elements) are preserved by Move (P).

The Laplace spectra are

$$\mathrm{Spec}_{\Delta}(D_2) = \left\{ \frac{15+\sqrt{33}}{2}, 5, \frac{15-\sqrt{33}}{2} \right\},$$
$$\mathrm{Spec}_{\Delta}(D_2^{(P)}) = \{13.3202, 9.86733, 7.4031, 4.96287, 4.44646\};$$

the adjacency spectra are

$$\mathrm{Spec}_{\mathbf{A}}(D_2) = \{2, 1, 1\},$$
$$\mathrm{Spec}_{\mathbf{A}}(D_2^{(P)}) = \{\approx 2.80194, \approx 1.44504, 1, 1, \approx -0.24698\};$$

the binary adjacency spectra are

$$\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_2) = \{1, 1, 1\},$$
$$\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_2^{(P)}) = \{1+\sqrt{2}, 1, 1, 1, 1-\sqrt{2}\};$$

the symmetric adjacency spectra are

$$\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_2) = \{\approx 6.15633, \approx 1.3691, \approx 0.474572\},$$
$$\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_2^{(P)}) = \{\approx 11.3293, \approx 4.32428, \approx 1.50561 \approx 0.824316, \approx 0.0164465\};$$

the symmetric binary adjacency spectra are

$$\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_2) = \{\approx 3.24698 \approx 1.55496, \approx 0.198062\},$$
$$\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_2^{(P)}) = \{\approx 7.45504, \approx 2.40912, \approx 1.52115, \approx 0.547884, \approx 0.0668084\};$$

the line adjacency spectra are

$$\mathrm{Spec}_{\mathbf{L}}(D_2) = \{2, 1, 1, 0, 0, 0\},$$
$$\mathrm{Spec}_{\mathbf{L}}(D_2^{(P)}) = \{\approx 2.80194, \approx 1.44504, 1, 1, \approx -0.24698, 0, 0, 0, 0, 0, 0, 0, 0, 0\};$$

the Hermitian adjacency spectra are

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{H}}(D_2) &= \left\{\sqrt{2}+1, 1, 1-\sqrt{2}\right\},\\
\mathrm{Spec}_{\mathbf{H}}(D_2^{(P)}) &= \left\{\frac{1}{2}\left(\sqrt{2\left(\sqrt{17}+5\right)}+2\right), \frac{1}{2}\left(\sqrt{2\left(5-\sqrt{17}\right)}+2\right),\right.\\
&\qquad \left.\frac{1}{2}\left(2-\sqrt{2\left(\sqrt{17}+5\right)}\right), 1, \frac{1}{2}\left(2-\sqrt{2\left(5-\sqrt{17}\right)}\right)\right\};
\end{aligned}$$

the skew adjacency spectra are

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{S}}(D_2) &= \left\{i\sqrt{2}, -i\sqrt{2}, 0\right\},\\
\mathrm{Spec}_{\mathbf{S}}(D_2^{(P)}) &= \left\{i\sqrt{\sqrt{5}+3}, -i\sqrt{\sqrt{5}+3}, i\sqrt{3-\sqrt{5}}, -i\sqrt{3-\sqrt{5}}, 0\right\};
\end{aligned}$$

the binary skew adjacency spectra are

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_2) &= \left\{i\sqrt{2}, -i\sqrt{2}, 0\right\},\\
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_2^{(P)}) &= \left\{i\sqrt{\frac{1}{2}\left(\sqrt{5}+3\right)}, -i\sqrt{\frac{1}{2}\left(\sqrt{5}+3\right)}, i\sqrt{\frac{1}{2}\left(3-\sqrt{5}\right)}, -i\sqrt{\frac{1}{2}\left(3-\sqrt{5}\right)}, 0\right\};
\end{aligned}$$

the skew Laplace spectra are

$$\begin{aligned}
\mathrm{Spec}_{\Delta_S}(D_2) &= \{i, -i, 0\},\\
\mathrm{Spec}_{\Delta_S}(D_2^{(P)}) &= \{-1, 1, 0, 0, 0\};
\end{aligned}$$

and the binary skew Laplace spectra are

$$\begin{aligned}
\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D_2) &= \{i, -i, 0\},\\
\mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D_2^{(P)}) &= \{0, 0, 0, 0, 0\}.
\end{aligned}$$

In Examples 5.1 and 5.2 the nonzero elements of the binary adjacency spectrum were preserved by Move (O). With the following, we indicate that this does not occur in general and also consider the spectra Spec_{Δ_N}, Spec_{Δ_C}, $\mathrm{Spec}_{\Delta_{\mathrm{b},N}}$, and $\mathrm{Spec}_{\Delta_{\mathrm{b},C}}$ that are only defined for strongly connected digraphs. We consider Move (O) as well as Move (I) for its relevance in Example 5.4 below.

Example 5.3. Let D_3 denote the digraph pictured in Figure 4A, and let $D_3^{(O)}$ denote the digraph obtained from D_3 by applying Move (O) to D_3 at vertex v_2 with the partition with two elements, the first containing the edge from v_2 to v_1 and a loop at v_2, and the second containing the other loop at v_2 (Figure 4B). Let $D_3^{(I)}$ denote the digraph obtained from D_3 by applying Move (I) to D_3 at vertex v_2 with the partition with two elements, the first containing the edge from v_1 to v_2 and a loop at v_2, and the second containing the other loop at v_2 (Figure 4C). Note that all three of these digraphs are strongly connected.

The binary adjacency spectra of these digraphs are

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_3) &= \left\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right\},\\
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_3^{(O)}) = \mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_3^{(I)}) &= \{1+\sqrt{2}, 1-\sqrt{2}, 0\},
\end{aligned}$$

demonstrating that Moves (O) and (I) do not preserve (the nonzero elements of) $\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}$.

The normalized Laplace spectra of these digraphs are

$$\begin{aligned}
\mathrm{Spec}_{\Delta_N}(D_3) &= \left\{\frac{4}{3}, 0\right\}, \\
\mathrm{Spec}_{\Delta_N}(D_3^{(O)}) &= \left\{\frac{1}{12}\left(2\sqrt{2}+13\right), \frac{1}{12}\left(13-2\sqrt{2}\right), 0\right\}, \\
\mathrm{Spec}_{\Delta_N}(D_3^{(I)}) &= \left\{\frac{1}{6}\left(\sqrt{3}+7\right), \frac{1}{6}\left(7-\sqrt{3}\right), 0\right\};
\end{aligned}$$

the combinatorial Laplace spectra of these digraphs are

$$\begin{aligned}
\mathrm{Spec}_{\Delta_C}(D_3) &= \left\{\frac{2}{3}, 0\right\}, \\
\mathrm{Spec}_{\Delta_C}(D_3^{(O)}) &= \left\{\frac{1}{12}\left(2\sqrt{3}+9\right), \frac{1}{12}\left(9-2\sqrt{3}\right), 0\right\}, \\
\mathrm{Spec}_{\Delta_C}(D_3^{(I)}) &= \left\{\frac{1}{6}\left(2\sqrt{3}+9\right), \frac{1}{6}\left(9-2\sqrt{3}\right), 0\right\};
\end{aligned}$$

the binary normalized Laplace spectra of these digraphs are

$$\begin{aligned}
\mathrm{Spec}_{\Delta_{\mathrm{b},N}}(D_3) &= \left\{\frac{3}{2}, 0\right\}, \\
\mathrm{Spec}_{\Delta_{\mathrm{b},N}}(D_3^{(O)}) &= \left\{\frac{1}{12}\left(2\sqrt{2}+13\right), \frac{1}{12}\left(13-2\sqrt{2}\right), 0\right\}, \\
\mathrm{Spec}_{\Delta_{\mathrm{b},N}}(D_3^{(I)}) &= \left\{\frac{1}{6}\left(\sqrt{3}+7\right), \frac{1}{6}\left(7-\sqrt{3}\right), 0\right\};
\end{aligned}$$

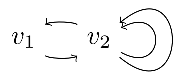
and the binary combinatorial Laplace spectra of these digraphs are

$$\begin{aligned}
\mathrm{Spec}_{\Delta_{\mathrm{b},C}}(D_3) &= \{1, 0\}, \\
\mathrm{Spec}_{\Delta_{\mathrm{b},C}}(D_3^{(O)}) &= \left\{\frac{1}{12}\left(2\sqrt{3}+9\right), \frac{1}{12}\left(9-2\sqrt{3}\right), 0\right\}, \\
\mathrm{Spec}_{\Delta_{\mathrm{b},C}}(D_3^{(I)}) &= \left\{\frac{1}{6}\left(2\sqrt{3}+9\right), \frac{1}{6}\left(9-2\sqrt{3}\right), 0\right\}.
\end{aligned}$$

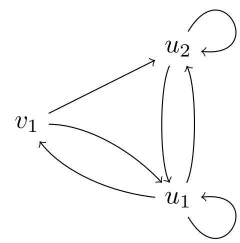
Example 5.4. Let D_4 denote the digraph used to illustrate Move (C) in [42, p.1204]; see Figure 5A. We apply moves (R), (O), (I), and (C) as described in Figure 5 and denote the resulting digraphs $D_4^{(R)}$, $D_4^{(O)}$, etc. Note that these digraphs are all strongly connected.

The Laplace spectra of these digraphs are

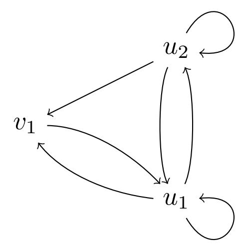
$$\begin{aligned}
\mathrm{Spec}_{\Delta}(D_4) &= \{2(2+\sqrt{2}), 2(2-\sqrt{2})\}, \\
\mathrm{Spec}_{\Delta}(D_4^{(R)}) &= \{8\}, \\
\mathrm{Spec}_{\Delta}(D_4^{(O)}) = \mathrm{Spec}_{\Delta}(D_4^{(I)}) &= \left\{\frac{7+\sqrt{33}}{2}, 5, \frac{7-\sqrt{33}}{2}\right\}, \\
\mathrm{Spec}_{\Delta}(D_4^{(C)}) &= \Bigg\{2\left(3+\sqrt{3+\sqrt{6}}\right), 2\left(3+\sqrt{3-\sqrt{6}}\right), \\
&\qquad 2\left(3-\sqrt{3-\sqrt{6}}\right), 2\left(3-\sqrt{3+\sqrt{6}}\right)\Bigg\},
\end{aligned}$$



(A) The digraph D_3.



(B) The digraph $D_3^{(O)}$ obtained by applying Move (O) to D_3 at v_2 using the partition whose first element contains the edge from v_2 to v_1 and a loop at v_2, and whose second element contains the other loop at v_2.

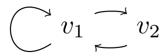


(C) The digraph $D_3^{(I)}$ obtained by applying Move (I) to D_3 at v_2 using the partition whose first element contains the edge from v_1 to v_2 and a loop at v_2, and whose second element contains the other loop at v_2.

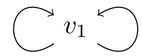
FIGURE 4. The strongly connected digraph D_3 and the application of Moves (O) and (I), also yielding strongly connected digraphs, from Example 5.3.

demonstrating that the nonzero elements of Spec_{Δ} are not preserved by any of these moves. The adjacency spectra are

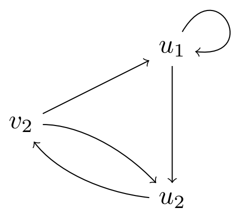
$$\begin{aligned}
\mathrm{Spec}_{\mathbf{A}}(D_4) &= \left\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right\}, \\
\mathrm{Spec}_{\mathbf{A}}(D_4^{(R)}) &= \{2\}, \\
\mathrm{Spec}_{\mathbf{A}}(D_4^{(O)}) = \mathrm{Spec}_{\mathbf{A}}(D_4^{(I)}) &= \left\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, 0\right\}, \\
\mathrm{Spec}_{\mathbf{A}}(D_4^{(C)}) &= \{\approx 2.53209, \approx 1.3473, \approx -0.879385, 0\},
\end{aligned}$$



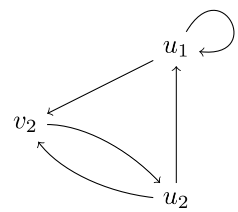
(A) The digraph D_4.



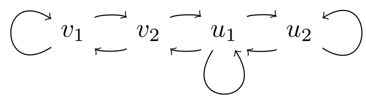
(B) The digraph $D_4^{(R)}$ obtained by applying Move (R).



(C) The digraph $D_4^{(O)}$.



(D) The digraph $D_4^{(I)}$.



(E) The digraph $D_4^{(C)}$ obtained from D_4 by applying Move (C) at vertex v_2.

FIGURE 5. The strongly connected digraph D_4 and the application of Moves (R), (O), (I), and (C), yielding strongly connected digraphs, from Example 5.4.

and the line adjacency spectra are

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{L}}(D_4) &= \left\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, 0\right\}, \\
\mathrm{Spec}_{\mathbf{L}}(D_4^{(R)}) &= \{2, 0\}, \\
\mathrm{Spec}_{\mathbf{L}}(D_4^{(O)}) = \mathrm{Spec}_{\mathbf{L}}(D_4^{(I)}) &= \left\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, 0, 0, 0\right\}, \\
\mathrm{Spec}_{\mathbf{L}}(D_4^{(C)}) &= \{\approx 2.53209, \approx 1.3473, \approx -0.879385, 0, 0, 0, 0, 0, 0\}.
\end{aligned}$$

Moves (O) and (I) preserve the nonzero elements of $\mathrm{Spec}_{\mathbf{A}}$ and $\mathrm{Spec}_{\mathbf{L}}$ by Propositions 3.4 and 4.2, while this example illustrates that Moves (R) and (C) do not. The binary adjacency spectrum of these digraphs are given by

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_4) &= \left\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right\}, \\
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_4^{(R)}) &= \{1\}, \\
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_4^{(O)}) = \mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_4^{(I)}) &= \left\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, 0\right\}, \\
\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}(D_4^{(C)}) &= \{\approx 2.53209, \approx 1.3473, \approx -0.879385, 0\}.
\end{aligned}$$

Hence, Moves (R) and (C) do not preserve the nonzero elements of $\mathrm{Spec}_{\mathbf{A}_{\mathrm{b}}}$ (while the same was demonstrated form Moves (O) and (I) in Example 5.3; see above).

The symmetric adjacency spectra are given by

$$\begin{aligned}
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_4) &= \left\{\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_4^{(R)}) &= \{4\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_4^{(O)}) = \mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_4^{(I)}) &= \{4,1,0\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}(D_4^{(C)}) &= \{\approx 6.41147, \approx 1.81521, \approx 0.773318, 0\};
\end{aligned}$$

the symmetric binary adjacency spectra are given by

$$\begin{aligned}
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_4) &= \left\{\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_4^{(R)}) &= \{1\},\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_4^{(O)}) = \mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_4^{(I)}) &= \{4,1,0\}\\
\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}(D_4^{(C)}) &= \{\approx 6.41147, \approx 1.81521, \approx 0.773318, 0\};
\end{aligned}$$

and the Hermitian adjacency spectra are given by

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{H}}(D_4) &= \left\{\frac{1}{2}\left(\sqrt{5}+1\right), \frac{1}{2}\left(1-\sqrt{5}\right)\right\},\\
\mathrm{Spec}_{\mathbf{H}}(D_4^{(R)}) &= \{1\},\\
\mathrm{Spec}_{\mathbf{H}}(D_4^{(O)}) = \mathrm{Spec}_{\mathbf{H}}(D_4^{(I)}) &= \left\{-\sqrt{3}, \sqrt{3}, 1\right\},\\
\mathrm{Spec}_{\mathbf{H}}(D_4^{(C)}) &= \{2.35567, 1.47726, -1.09529, 0.26236\},
\end{aligned}$$

demonstrating that the nonzero elements of $\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}}$, $\mathrm{Spec}^{\mathcal{S}}_{\mathbf{A}_{\mathrm{b}}}$, and $\mathrm{Spec}_{\mathbf{H}}$ are not preserved by these four moves.

The skew adjacency spectra are given by

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{S}}(D_4) &= \{0,0\},\\
\mathrm{Spec}_{\mathbf{S}}(D_4^{(R)}) &= \{0\},\\
\mathrm{Spec}_{\mathbf{S}}(D_4^{(O)}) = \mathrm{Spec}_{\mathbf{S}}(D_4^{(I)}) &= \left\{i\sqrt{2}, -i\sqrt{2}, 0\right\},\\
\mathrm{Spec}_{\mathbf{S}}(D_4^{(C)}) &= \{0,0,0,0\};
\end{aligned}$$

the binary skew adjacency spectra are given by

$$\begin{aligned}
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_4) &= \{0,0\},\\
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_4^{(R)}) &= \{0\},\\
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_4^{(O)}) = \mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_4^{(I)}) &= \left\{i\sqrt{2}, -i\sqrt{2}, 0\right\},\\
\mathrm{Spec}_{\mathbf{S}_{\mathrm{b}}}(D_4^{(C)}) &= \{0,0,0,0\};
\end{aligned}$$

the skew Laplace spectra are given by

$$\begin{aligned}
\mathrm{Spec}_{\Delta_S}(D_4) &= \{0,0\},\\
\mathrm{Spec}_{\Delta_S}(D_4^{(R)}) &= \{0\},\\
\mathrm{Spec}_{\Delta_S}(D_4^{(O)}) = \mathrm{Spec}_{\Delta_S}(D_4^{(I)}) &= \{i,-i,0\},\\
\mathrm{Spec}_{\Delta_S}(D_4^{(C)}) &= \{0,0,0,0\};
\end{aligned}$$

and the binary skew Laplace spectra are given by

$$\begin{aligned}
\operatorname{Spec}_{\Delta_{\mathrm{b},S}}(D_4) &= \{0, 0\}, \\
\operatorname{Spec}_{\Delta_{\mathrm{b},S}}(D_4^{(R)}) &= \{0\}, \\
\operatorname{Spec}_{\Delta_{\mathrm{b},S}}(D_4^{(O)}) = \operatorname{Spec}_{\Delta_{\mathrm{b},S}}(D_4^{(I)}) &= \{i, -i, 0\}, \\
\operatorname{Spec}_{\Delta_{\mathrm{b},S}}(D_4^{(C)}) &= \{0, 0, 0, 0\},
\end{aligned}$$

demonstrating that the nonzero elements of $\operatorname{Spec}_{\mathbf{S}}$, $\operatorname{Spec}_{\mathbf{S}_{\mathrm{b}}}$, $\operatorname{Spec}_{\Delta_S}$, and $\operatorname{Spec}_{\Delta_{\mathrm{b},S}}$ are not preserved by Moves (O) nor (I). Note that Move (C) preserves these the nonzero elements of these spectra by Proposition 4.5; the case of Move (R) will be considered in Example 5.5 below.

The normalized Laplace spectra are given by

$$\begin{aligned}
\operatorname{Spec}_{\Delta_N}(D_4) &= \left\{\frac{3}{2}, 0\right\}, \\
\operatorname{Spec}_{\Delta_N}(D_4^{(R)}) &= \{0\}, \\
\operatorname{Spec}_{\Delta_N}(D_4^{(O)}) = \operatorname{Spec}_{\Delta_N}(D_4^{(I)}) &= \left\{\frac{7}{4}, \frac{3}{4}, 0\right\}, \\
\operatorname{Spec}_{\Delta_N}(D_4^{(C)}) &= \left\{\frac{3}{2}, \frac{1}{12}\left(\sqrt{13}+7\right), \frac{1}{12}\left(7-\sqrt{13}\right), 0\right\};
\end{aligned}$$

the combinatorial Laplace spectra are given by

$$\begin{aligned}
\operatorname{Spec}_{\Delta_C}(D_4) &= \{2, 0\}, \\
\operatorname{Spec}_{\Delta_C}(D_4^{(R)}) &= \{0\}, \\
\operatorname{Spec}_{\Delta_C}(D_4^{(O)}) = \operatorname{Spec}_{\Delta_C}(D_4^{(I)}) &= \left\{\frac{7}{4}, \frac{3}{4}, 0\right\}, \\
\operatorname{Spec}_{\Delta_C}(D_4^{(C)}) &= \left\{\frac{1}{2}\left(\sqrt{2}+2\right), 1, \frac{1}{2}\left(2-\sqrt{2}\right), 0\right\};
\end{aligned}$$

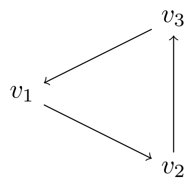
the binary normalized Laplace spectra are given by

$$\begin{aligned}
\operatorname{Spec}_{\Delta_{\mathrm{b},N}}(D_4) &= \left\{\frac{3}{2}, 0\right\}, \\
\operatorname{Spec}_{\Delta_{\mathrm{b},N}}(D_4^{(R)}) &= \{0\}, \\
\operatorname{Spec}_{\Delta_{\mathrm{b},N}}(D_4^{(O)}) = \operatorname{Spec}_{\Delta_{\mathrm{b},N}}(D_4^{(I)}) &= \left\{\frac{7}{4}, \frac{3}{4}, 0\right\}, \\
\operatorname{Spec}_{\Delta_{\mathrm{b},N}}(D_4^{(C)}) &= \left\{\frac{3}{2}, \frac{1}{12}\left(\sqrt{13}+7\right), \frac{1}{12}\left(7-\sqrt{13}\right), 0\right\};
\end{aligned}$$

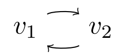
and the binary combinatorial Laplace spectra are given by

$$\begin{aligned}
\operatorname{Spec}_{\Delta_{\mathrm{b},C}}(D_4) &= \{2, 0\}, \\
\operatorname{Spec}_{\Delta_{\mathrm{b},C}}(D_4^{(R)}) &= \{0\}, \\
\operatorname{Spec}_{\Delta_{\mathrm{b},C}}(D_4^{(O)}) = \operatorname{Spec}_{\Delta_{\mathrm{b},C}}(D_4^{(I)}) &= \left\{\frac{7}{4}, \frac{3}{4}, 0\right\}, \\
\operatorname{Spec}_{\Delta_{\mathrm{b},C}}(D_4^{(C)}) &= \left\{\frac{1}{2}\left(\sqrt{2}+2\right), 1, \frac{1}{2}\left(2-\sqrt{2}\right), 0\right\},
\end{aligned}$$

demonstrating that none of these moves preserve the nonzero elements of $\operatorname{Spec}_{\Delta_N}$, $\operatorname{Spec}_{\Delta_C}$, $\operatorname{Spec}_{\Delta_{\mathrm{b},N}}$, nor $\operatorname{Spec}_{\Delta_{\mathrm{b},C}}$.

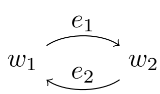


(A) The digraph D_5.

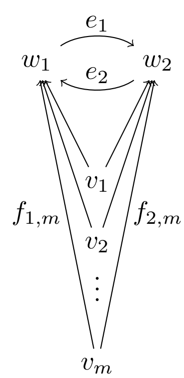


(B) The digraph $D_5^{(R)}$ obtained from D_5 by applying Move (R) at vertex v_1.

Figure 6. The strongly connected digraph D_5 and the application of Move (R), yielding a strongly connected digraph, from Example 5.5.



(A) The digraph D_0.



(B) The digraph D_m.

Figure 7. The digraphs D_0 and D_m ofrom Example 5.6.

It remains only to demonstrate that Move (R) does not preserve the nonzero elements of $\mathrm{Spec}_{\mathbf{S}}$, $\mathrm{Spec}_{\mathbf{S}_\mathrm{b}}$, Spec_{Δ_S}, nor $\mathrm{Spec}_{\Delta_{\mathrm{b},S}}$.

Example 5.5. Let D_5 denote the digraph in Figure 6A given by a single cycle of length 3, and let $D_5^{(R)}$ denote the result of applying Move (R) to D_5 at vertex v_1. Note that both D_5 and $D_5^{(R)}$ are strongly connected.

The skew adjacency spectra, binary skew adjacency spectra, skew Laplace spectra, and binary skew Laplace spectra of these digraphs all coincide and are given by

$$\mathrm{Spec}_{\mathbf{S}}(D_5) = \mathrm{Spec}_{\mathbf{S}_\mathrm{b}}(D_5) = \mathrm{Spec}_{\Delta_S}(D_5) = \mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D_5) = \{i\sqrt{3}, -i\sqrt{3}, 0\}$$
$$\mathrm{Spec}_{\mathbf{S}}(D_5^{(R)}) = \mathrm{Spec}_{\mathbf{S}_\mathrm{b}}(D_5^{(R)}) = \mathrm{Spec}_{\Delta_S}(D_5^{(R)}) = \mathrm{Spec}_{\Delta_{\mathrm{b},S}}(D_5^{(R)}) = \{0, 0\}.$$

Hence, Move (R) does not preserve the nonzero elements of any of these spectra.

We end this section with the following example, which indicates the degree to which several features of the Laplace spectrum can change within a class of digraphs with Morita equivalent C^*-algebras. Specifically, repeated application of Move (S) yields a family of digraphs for which the number of nonzero elements of Spec_Δ, the maximum multiplicity of a nonzero element of Spec_Δ, and the maximum modulus of an element of Spec_Δ are all unbounded.

Example 5.6. Let D_0 be the digraph with $V_0 = \{w_1, w_2\}$ and $E_0 = \{e_1, e_2\}$, where $s(e_1) = w_1$, $s(e_2) = w_2$, $r(e_1) = w_2$, and $r(e_2) = w_1$; see Figure 7A. Let D_m be the digraph with vertices $V_m = V_0 \cup \{v_i : i = 1, \dots, m\}$ and edges $E_m = E_0 \cup \{f_{1,i}, f_{2,i} : i = 1, \dots, m\}$, where for each i, $s(f_{1,i}) = s(f_{2,i}) = v_i$, $r(f_{1,i}) = w_1$, and $r(f_{2,i}) = w_2$; see Figure 7B. Then applying move S to remove vertex v_n from D_m yields D_{m-1} for $m \geqslant 1$.

Ordering the vertices and edges of D_m as listed, we have

$$\mathbf{M}(D_0) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Delta D_0 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad \mathrm{Spec}_\Delta(D_0) = \{4, 0\}.$$

Ordering the vertices and edges of D_m as listed, we have

$$\mathbf{M}(D) = \begin{pmatrix} \mathbf{M}(D_0) & I_2 & I_2 & \cdots & I_2 \\ \mathbf{0} & b & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & b & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & b \end{pmatrix},$$

where I_2 is the 2×2 identity matrix, each $\mathbf{0}$ is a 1×2 block, and b is the 1×2 block $(-1, -1)$. Then

$$\Delta D_m = \begin{pmatrix} \begin{pmatrix} m+2 & 2 \\ 2 & m+2 \end{pmatrix} & B & B & \cdots & B \\ B & 2I_2 & \mathbf{0} & \cdots & \mathbf{0} \\ B & \mathbf{0} & 2I_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & \mathbf{0} & \cdots & \mathbf{0} & 2I_2 \end{pmatrix},$$

where $\mathbf{0}$ is a 2×2 block B is a 2×2 block with all entries -1. Hence

$$\mathrm{Spec}_\Delta(D_m) = \{m+4, m+2, \overbrace{2, 2, \ldots, 2}^{m-1 \text{ times}}, 0\}.$$

Hence, among the D_m, $m \geqslant 0$, the maximum multiplicity and maximum modulus of an element of $\mathrm{Spec}_\Delta(D_m)$ are unbounded.

6. The case of k-graphs

In this section, we briefly discuss the extension of the results of this paper to the case of finite k-graphs. Recall [33, Definition 1.1] that a *k-graph* $(\Lambda, \deg)$ is a countable small category Λ equipped with a functor $\deg\colon \Lambda \to \mathbb{N}^k$ satisfying a factorization property: if $\deg(\lambda) = m + n$ for $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$, then λ can be factored uniquely as $\lambda = \mu\nu$ with $\deg(\mu) = m$ and $\deg(\nu) = n$. As explained in [33, Example 1.3], there is a correspondence between 1-graphs and digraphs, where the category associated to a digraph is the so-called *path category*, whose objects are vertices, morphisms are paths of the digraph, and the degree functor is given by the path length. Hence, k-graphs form a generalization of digraphs, and in [33] a C^*-algebra is associated to a k-graph in such a way that the C^*-algebra of a 1-graph coincides with the C^*-algebra of the corresponding digraph.

The recent paper [16] introduced generalizations to row-finite k-graphs of the moves recalled in Section 4 above and proved that they preserve the Morita equivalence class of the corresponding C^*-algebra, making progress towards a complete description of the equivalence class of k-graphs with Morita equivalent C^*-algebras.

Note that using the results of [28], a k-graph can be thought of as a digraph with colored edges and an equivalence relation on the set of paths, where the colors correspond to the standard basis vectors of $\mathbb{N}^k$; see [28] or [16, Section 2] for more information. If $(\Lambda, \deg)$ is a k-graph, we refer to the corresponding colored digraph as the *underlying colored digraph* of $(\Lambda, \deg)$. Similarly, the ordinary digraph formed by forgetting the coloring of the edges of the underlying colored digraph is the *underlying (uncolored) digraph*. The moves introduced in [16] are described as moves on the underlying colored digraph. As well, each of the notions of the spectrum of a digraph given in Definition 2.8 admit an extension to the case of a k-graph by defining the spectrum of the k-graph to be the spectrum of its underlying uncolored digraph; this coincides with summing the matrix used to define the spectrum over the colors as was done for the Laplacian in [23]. Note that in the context of k-graphs, a digraph is often replaced with its transpose, the digraph formed by reversing the direction of each edge. Hence, in-splitting a

k-graph corresponds to out-splitting the underlying uncolored digraph, the analogue for k-graphs of Move (S) is the deletion of a sink, etc.

The move of *in-splitting a k-graph* [16, Section 3] is an analogue of Move (O) for digraphs. Specifically, applying this move to a k-graph corresponds exactly to applying Move (O) to the underlying uncolored digraph, where the partition of the uncolored graph is the same partition used to in-split with the colors forgotten. Hence, in-splitting a k-graph preserves the multiset of nonzero elements of the adjacency spectrum and the multiset of nonzero elements of the line adjacency spectrum, while none of the other spectra are preserved.

The move of *delaying a k-graph* [16, Section 4] is an analogue of the inverse of Move (R), though is much more restricted and involved in the context of k-graphs. In examples, applying the delay move to the k-graph does not preserve any of the spectra we consider.

Sink deletion [16, Section 5] is an analogue of Move (S), though it does not necessarily correspond to applying Move (S) to the underlying uncolored digraph. Specifically, for the underlying colored graph of a k-graph, a vertex can be deleted if it is a sink with respect to the edges of one color but not the others. Hence, the deleted vertex need not be a sink (or source) of the underlying uncolored digraph. Hence, sink deletion preserves the multisets of nonzero elements of the adjacency spectrum, the line adjacency spectrum, and the binary adjacency spectrum only when applied to a vertex that is simultaneously a sink in every color. We have observed in examples that these spectra are not preserved when this move is applied to a vertex that is not a sink in every color.

Reduction [16, Section 6] for k-graphs corresponds to applying Move (R) to the underlying uncolored digraph; though the inverse of any delay operation is a reduction for (uncolored) digraphs, the relationship in the case of k-graphs is more subtle. As in the case of digraphs, reduction of k-graphs does not preserve any of the spectra in examples.

References

1. Stephen Allen, David Pask, and Aidan Sims, *A dual graph construction for higher-rank graphs, and K-theory for finite 2-graphs*, Proc. Amer. Math. Soc. **134** (2006), no. 2, 455–464. MR 2176014
2. Teresa Bates and David Pask, *Flow equivalence of graph algebras*, Ergodic Theory Dynam. Systems **24** (2004), no. 2, 367–382. MR 2054048
3. Norman Biggs, *Algebraic graph theory*, Cambridge University Press, London, 1974, Cambridge Tracts in Mathematics, No. 67. MR 0347649
4. R. Bowen and O. E. Lanford, III, *Zeta functions of restrictions of the shift transformation*, Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 43–49. MR 0271401
5. Richard A. Brualdi, *Spectra of digraphs*, Linear Algebra Appl. **432** (2010), no. 9, 2181–2213. MR 2599853
6. Richard A. Brualdi and Herbert J. Ryser, *Combinatorial matrix theory*, Encyclopedia of Mathematics and its Applications, vol. 39, Cambridge University Press, Cambridge, 1991. MR 1130611
7. Steven Kay Butler, *Eigenvalues and structures of graphs*, ProQuest LLC, Ann Arbor, MI, 2008, Thesis (Ph.D.)–University of California, San Diego. MR 2711548
8. M. Cavers, S. M. Cioabă, S. Fallat, D. A. Gregory, W. H. Haemers, S. J. Kirkland, J. J. McDonald, and M. Tsatsomeros, *Skew-adjacency matrices of graphs*, Linear Algebra Appl. **436** (2012), no. 12, 4512–4529. MR 2917427
9. Fan Chung, *Spectral graph theory*, CBMS Regional Conference Series in Mathematics, vol. 92, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997. MR 1421568
10. ———, *Laplacians and the Cheeger inequality for directed graphs*, Ann. Comb. **9** (2005), no. 1, 1–19. MR 2135772
11. ———, *The diameter and Laplacian eigenvalues of directed graphs*, Electron. J. Combin. **13** (2006), no. 1, Note 4, 6. MR 2212511
12. Yves Colin de Verdière, *Sur la multiplicité de la première valeur propre non nulle du laplacien*, Comment. Math. Helv. **61** (1986), no. 2, 254–270. MR 856089
13. J. Cuntz, *A class of C^*-algebras and topological Markov chains. II. Reducible chains and the Ext-functor for C^*-algebras*, Invent. Math. **63** (1981), no. 1, 25–40. MR 608527
14. Joachim Cuntz and Wolfgang Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), no. 3, 251–268. MR 561974
15. Harm Derksen and Gregor Kemper, *Computational invariant theory*, enlarged ed., Encyclopaedia of Mathematical Sciences, vol. 130, Springer, Heidelberg, 2015, With two appendices by Vladimir L. Popov, and an addendum by Norbert A'Campo and Popov, Invariant Theory and Algebraic Transformation Groups, VIII. MR 3445218

16. Caleb Eckhardt, Kit Fieldhouse, Daniel Gent, Elizabeth Gillaspy, Ian Gonzales, and David Pask, *Moves on k-graphs preserving morita equivalence*, `arXiv:2006.13441 [math.OA]`, 2020.
17. Michela Egidi and Olaf Post, *Asymptotic behaviour of the Hodge Laplacian spectrum on graph-like manifolds*, J. Spectr. Theory **7** (2017), no. 2, 433–469. MR 3662014
18. Søren Eilers, Gunnar Restorff, and Efren Ruiz, *On graph C^*-algebras with a linear ideal lattice*, Bull. Malays. Math. Sci. Soc. (2) **33** (2010), no. 2, 233–241. MR 2666426
19. Søren Eilers, Gunnar Restorff, Efren Ruiz, and Adam P. W. Sørensen, *Invariance of the Cuntz splice*, Math. Ann. **369** (2017), no. 3-4, 1061–1080. MR 3713535
20. ———, *Geometric classification of graph* C*-*algebras over finite graphs*, Canad. J. Math. **70** (2018), no. 2, 294–353. MR 3759003
21. ———, *The complete classification of unital graph c*-algebras: Geometric and strong*, `arXiv:1611.07120v2 [math.OA]` , 2019.
22. Søren Eilers, Efren Ruiz, and Adam P. W. Sørensen, *Amplified graph C^*-algebras*, Münster J. Math. **5** (2012), 121–150. MR 3047630
23. Carla Farsi, Elizabeth Gillaspy, Sooran Kang, and Judith Packer, *Wavelets and graph C^*-algebras*, Excursions in harmonic analysis. Vol. 5, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 2017, pp. 35–86. MR 3699678
24. Carla Farsi, Emily Proctor, and Christopher Seaton, *Approximating orbifold spectra using collapsing connected sums*, `arXiv:1611.07676v2 [math.DG]`, 2020.
25. Hilal A. Ganie, *Bounds for the skew Laplacian (skew adjacency) spectral radius of a digraph*, Trans. Comb. **8** (2019), no. 2, 1–12. MR 3950979
26. Krystal Guo and Bojan Mohar, *Hermitian adjacency matrix of digraphs and mixed graphs*, J. Graph Theory **85** (2017), no. 1, 217–248. MR 3634484
27. Frank Harary, Clarence King, Abbe Mowshowitz, and Ronald C. Read, *Cospectral graphs and digraphs*, Bull. London Math. Soc. **3** (1971), 321–328. MR 294176
28. Robert Hazlewood, Iain Raeburn, Aidan Sims, and Samuel B. G. Webster, *Remarks on some fundamental results about higher-rank graphs and their C^*-algebras*, Proc. Edinb. Math. Soc. (2) **56** (2013), no. 2, 575–597. MR 3056660
29. Wolfram Research, Inc., *Mathematica, Version 12.1*, Champaign, IL, 2020.
30. Pierre Jammes, *Prescription de la multiplicité des valeurs propres du laplacien de Hodge–de Rham*, Comment. Math. Helv. **86** (2011), no. 4, 967–984. MR 2851874
31. Irena M. Jovanović, *Non-negative spectrum of a digraph*, Ars Math. Contemp. **12** (2017), no. 1, 167–182. MR 3610772
32. V. Krishnamoorthy and K. R. Parthasarathy, *Cospectral graphs and digraphs with given automorphism group*, J. Combinatorial Theory Ser. B **19** (1975), no. 3, 204–213. MR 398884
33. Alex Kumjian and David Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20. MR 1745529
34. Bolian Liu and Hong-Jian Lai, *Matrices in combinatorics and graph theory*, Network Theory and Applications, vol. 3, Kluwer Academic Publishers, Dordrecht, 2000, With a foreword by Richard A. Brualdi. MR 1795558
35. Jianxi Liu and Xueliang Li, *Hermitian-adjacency matrices and Hermitian energies of mixed graphs*, Linear Algebra Appl. **466** (2015), 182–207. MR 3278246
36. I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015, With contribution by A. V. Zelevinsky and a foreword by Richard Stanley, Reprint of the 2008 paperback edition [MR1354144]. MR 3443860
37. M. H. Mann, Iain Raeburn, and C. E. Sutherland, *Representations of compact groups, Cuntz-Krieger algebras, and groupoid C^*-algebras*, Miniconference on probability and analysis (Sydney, 1991), Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 29, Austral. Nat. Univ., Canberra, 1992, pp. 135–144. MR 1188891
38. Bojan Mohar, *Hermitian adjacency spectrum and switching equivalence of mixed graphs*, Linear Algebra Appl. **489** (2016), 324–340. MR 3421853
39. ———, *Hermitian adjacency spectrum and switching equivalence of mixed graphs*, Linear Algebra Appl. **489** (2016), 324–340. MR 3421853
40. Gunnar Restorff, *Classification of Cuntz-Krieger algebras up to stable isomorphism*, J. Reine Angew. Math. **598** (2006), 185–210. MR 2270572
41. Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR 924157
42. Adam P. W. Sørensen, *Geometric classification of simple graph algebras*, Ergodic Theory Dynam. Systems **33** (2013), no. 4, 1199–1220. MR 3082546
43. Richard P. Stanley, *Algebraic combinatorics*, Undergraduate Texts in Mathematics, Springer, Cham, 2018, Walks, trees, tableaux, and more, Second edition of [MR3097651]. MR 3823204
44. Christopher Storm, *An iterative construction of isospectral digraphs*, Discrete Math. **311** (2011), no. 14, 1323–1332. MR 2795542
45. Pepijn Wissing and Edwin R. van Dam, *The negative tetrahedron and the first infinite family of connected digraphs that are strongly determined by the Hermitian spectrum*, J. Combin. Theory Ser. A **173** (2020), 105232, 29. MR 4067471

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