Γ-EXTENSIONS OF THE SPECTRUM OF AN ORBIFOLD

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ABSTRACT. We introduce the Γ-extension of the spectrum of the Laplacian of a Riemannian orbifold, where Γ is a finitely generated discrete group. This extension, called the Γ-*spectrum,* is the union of the Laplace spectra of the Γ-sectors of the orbifold, and hence constitutes a Riemannian invariant that is directly related to the singular set of the orbifold. We compare the Γ-spectra of known examples of isospectral pairs and families of orbifolds and demonstrate that in many cases, isospectral orbifolds need not be Γ-isospectral. We additionally prove a version of Sunada's theorem that allows us to construct pairs of orbifolds that are Γ-isospectral for any choice of Γ.

CONTENTS

1. INTRODUCTION

A central question of spectral geometry concerns the extent to which the Laplace spectrum of an orbifold influences its geometry and topology, and vice versa. For example, if two orbifolds have the same spectrum, they must have the same volume and dimension [13]. On the other hand, there are many examples of isospectal, nonisometric orbifolds. Most of the early examples of isospectral orbifolds were manifolds [23, 37, 19, 33, 17, 29]. More recently, however, attention has turned to the study of the spectrum of orbifolds having nontrivial singular sets and the interplay between the spectrum and the singular set. In 2006, Shams, Stanhope, and Webb produced arbitrarily large finite families of isospectral orbifolds [31]. Any pair

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of orbifolds in a given family contains points with nonisomorphic isotropy groups, but all orbifolds in the family have maximal isotropy groups of the same order. Rossetti, Schueth, and Weilandt have since produced examples of pairs isospectral orbifolds having different maximal isotropy order [28]. Working independently, both Sutton, and the second author working with Stanhope, found examples of continuous families of isospectral, nonisometric orbifolds [34, 26]. More recently, Shams produced examples of pairs of isospectral, nonisometric orbifold lens spaces [30]. In the positive direction, Dryden, Gordon, Greenwald, and Webb constructed the asymptotic expansion for the heat trace of a general compact Riemannian orbifold in such a way that the contribution of each piece of the singular set to the heat invariants is evident [9]. They used their results to show that the spectrum can distinguish orbifolds within certain classes of two-dimensional orbifolds. Dryden and Strohmaier showed that for a compact orientable hyperbolic orbisurface, the numbers and types of singular points as well as the length spectrum of the orbifold are completely determined by the Laplace spectrum [10]. This was shown independently by Doyle and Rossetti, who in [8] also proved an extension to the case of compact hyperbolic orbisurfaces that are not necessarily connected or orientable. In [25], the second author proved that any isospectral collection of orbifolds with sectional curvature uniformly bounded below and having only isolated singular points contains only finitely many orbifold category homeomorphism types.

In this paper, we address the question of the relationship between the spectrum of an orbifold O and its singular set by introducing the Γ*-spectrum* of O. The Γ-spectrum is the Γ-extension of the spectrum of the Laplacian in the sense of [36], meaning that it is an application of the spectrum to the Γ*-sectors* of O. Originally introduced in [35] for global quotients and [15] for general orbifolds, the Γ-sectors of O consist of a disjoint union of orbifolds of varying dimensions including a copy of O as well as other components that finitely cover the singular set of O. Special cases include the *inertia orbifold* when $\Gamma = \mathbb{Z}$ and the *multisectors* when Γ is a free group; see e.g. [1]. The orbifold of Γ-sectors can be thought of an "unraveling" of the singular set of O into distinct orbifolds, where the group Γ determines the type of singularities that are unraveled. In this sense, the Γ-spectrum includes the ordinary spectrum of O as well as, approximately speaking, the spectra of various components of the singular locus of O. The technique of extending an orbifold invariant by considering its Γ-extension has been studied in the case of Euler characteristics and related orbifold invariants; here, we apply this technique to the case of an invariant of a Riemannian orbifold, the spectrum of the Laplacian.

The definition of an orbifold varies considerably from author to author and discipline to discipline, based on the features of the orbifold structure that are under consideration. In particular, in Riemannian geometry, orbifolds are usually assumed to be effective, and hence can be presented as quotient orbifolds; see Section 2. Considering the Γ-spectrum of a noneffective orbifold leads to trivial examples that are contrary to the spirit of the investigation presented here; see Section 2.2 and in particular Examples 2.7 and 2.8. Hence, we consider the Γ-spectrum to be most interesting when applied to an effective Riemannian orbifold. For this reason, though we do explain the (direct) generalizations of the definitions presented here to general orbifolds presented by groupoids, we otherwise restrict our attention to quotient orbifolds. Note that the Γ-sectors of a nontrivial orbifold O may include noneffective orbifolds even when O is assumed effective, and thus our discussion requires the consideration of noneffective orbifolds. If O is a quotient orbifold, however, the Γ-sectors of O are quotient orbifolds as well, even when they are not effective.

This paper is organized as follows. In Section 2, we recall the relevant definitions and fix notation. The Γ-sectors are defined in Section 2.1, and the Γ-spectrum is defined in Section 2.2. In Section 2.3, we discuss some immediate consequences of the definition of the Γ-spectrum. Section 3 contains a description of the Γ-sectors of several collections of known isospectral, nonisometric orbifolds in order to determine whether they are also Γ-isospectral. In Section 4, we prove a Sunada-type theorem for Γ-isospectrality and exhibit examples of nonisometric orbifolds that are nontrivially Γ-isospectral for different choices of Γ.

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2. Background and definitions

Let O be an n-dimensional orbifold. We will be primarily interested in the case that O can be presented as a *quotient orbifold*, i.e. O is given by $G\backslash M$ with orbifold structure given by the translation groupoid $G \ltimes M$ where M is a smooth manifold and G is a compact Lie group acting with finite isotropy groups (on the left) on M. If G is finite, then we say O is a *global quotient orbifold.* More generally, an orbifold is defined by a Morita equivalence class of a proper étale Lie groupoid; see [1, Definition 1.48] and the following paragraph. A proper étale Lie groupoid can be thought of as an atlas for the orbifold it represents, and the orbifold is given by the orbit space of the groupoid under the action of its arrows. In either case, O consists of a second countable Hausdorff space $\mathbb{X}_O$, the *underlying space of* O, that is covered by *orbifold charts* of the form $\{V_x, G_x, \pi_x\}$ where V_x is diffeomorphic to $\mathbb{R}^n$, G_x is a finite group acting linearly on V_x, and $\pi_x\colon V_x \to \mathbb{X}_O$ induces a homeomorphism of $G_x\backslash V_x$ onto an open subset of $\mathbb{X}_O$. If O is *effective*, i.e. for each orbifold chart $\{V_x, G_x, \pi_x\}$ the group G_x acts effectively on V_x, then it is well known that O can be presented as a quotient orbifold using the *frame bundle construction*, and that G can be taken to be $O(n)$; see [1, Theorem 1.23]. It has been conjectured that all orbifolds can be presented as quotient orbifolds; see [1, p. 27].

Equivalence of orbifolds is subtle and most easily described in terms of Morita equivalence of groupoid presentations. Two groupoids $\mathcal{G}$ and $\mathcal{G}'$ are *Morita equivalent* if there is a groupoid $\mathcal{H}$ and a chain of groupoid equivalences $\mathcal{G} \leftarrow \mathcal{H} \rightarrow \mathcal{G}'$. (See [1, Definition 1.42] for the definition of groupoid equivalence.) Each groupoid equivalence induces a homeomorphism between underlying spaces of the orbifolds that the groupoids represent, preserving the isomorphism class of the isotropy group of each point in the orbifold. By identifying the underlying spaces via these homeomorphisms, one may think of $\mathcal{H}$ as an orbifold atlas that refines the orbifold atlases corresponding to $\mathcal{G}$ and $\mathcal{G}'$. Note in particular that for two groupoids $\mathcal{G}$ and $\mathcal{G}'$ to be Morita equivalent, there need not be a map directly from $\mathcal{G}$ to $\mathcal{G}'$. Intuitively, this corresponds to the fact that the atlas corresponding to $\mathcal{G}$ might not be fine enough to locally define a map.

If O and O' are quotient orbifolds represented by $G \ltimes M$ and $H \ltimes N$ respectively, then an *equivariant map* from O to O' consists of a homomorphism $\varphi : G \to H$ and a smooth map $f : M \to N$ such that $f(gx) = \varphi(g)f(x)$ for all $g \in G$ and $x \in M$. It has been shown that if we restrict our attention to the category of smooth translation groupoids $G \ltimes M$ and equivariant maps, then Morita equivalence is still a well-defined equivalence relation on this category [27]. Thus when we restrict our attention to orbifolds that can be represented as quotients, we need only concern ourselves with equivariant groupoid equivalences, and with Morita equivalences via smooth translation groupoids. Hence we define a *diffeomorphism* between quotient orbifolds to be an equivariant groupoid equivalence. We note that by Proposition 3.5 in [27], every diffeomorphism between quotient orbifolds is given as a composition of quotient and inductive groupoid equivalences. We say that two quotient orbifolds O and O' represented by $G \ltimes M$ and $H \ltimes N$ respectively are *diffeomorphic* if they can be connected by a Morita equivalence

$$G \ltimes M \xleftarrow{(\varphi,f)} K \ltimes Z \xrightarrow{(\omega,h)} H \ltimes N$$

where (φ, f) and (ω, h) are equivariant groupoid equivalences.

Classically, a Riemannian metric on an orbifold O has been defined via charts. For each chart $\{V_x, G_x, \pi_x\}$, let g_x be a G_x-invariant Riemannian metric on V_x. Patching the charts together via a partition of unity gives a *Riemannian structure* on O. If O is presented as a quotient $G \ltimes M$ for G a compact Lie group, then any G-invariant metric on M induces a metric on the orbifold O, and any metric on O is induced by such a metric; see [32, Proposition 2.1]. In particular, given a point $x \in M$, a slice W_x at x for the G-action on M induces a local chart $\{W_x, G_x, \pi_x\}$ for the orbifold O at Gx; see [12, Definition 2.3.1]. The G-invariant metric on M restricts to a G_x-invariant metric on W_x. Note that distinct G-invariant metrics on M may correspond to the same metric on O.

When studying the Riemannian geometry of orbifolds, it is common and natural to restrict to the consideration of effective orbifolds. Indeed, the Riemannian structure of a non-effective orbifold O is identical to that of its *effectivisation* O_{eff}; see [1, Definition 2.33], except for a minor change to integration on the orbifold, see Section 2.3. If O is a noneffective orbifold with quotient presentation $G \ltimes M$ and K is the (necessarily normal) subgroup of G that acts trivially on M, then O_{eff} is the effective orbifold presented by $G/K \ltimes M$, and it is clear that the Riemiannian structures of O and O_{eff} coincide. However, when considering the Γ-sectors and Γ-spectrum, we will see that a noneffective group action may arise in the Γ-sectors, even in the case that O is effective. For this reason, we make the following.

Definition 2.1. Let $O = (G \ltimes M, g)$ and $O' = (H \ltimes N, g')$ be Riemannian quotient orbifolds. Let $\bar{g}$ and $\bar{g}'$ be correponding invariant metrics on M and N respectively. An *isometry* from O to O' is an equivariant groupoid equivalence $(\varphi, f) : G \ltimes M \to H \ltimes N$ such that $f^*\bar{g}' = \bar{g}$. We say that O and O' are *isometric* if they can be connected by a chain of isometries $G \ltimes M \xleftarrow{(\varphi,f)} K \ltimes Z \xrightarrow{(\omega,h)} H \ltimes N$. If O_{eff} and O'_{eff} are isometric, we say that O and O' are *effectively isometric*.

For instance, any Riemannian manifold M may be equipped with the trivial action of a finite group G resulting in the noneffective orbifold O presented by $G \ltimes M$. The orbifolds M and O are identical in every sense that is significant to Riemannian geometry, and hence are effectively isometric. However, because the isotropy group of each point in an orbifold is invariant under diffeomorphism, they are not isometric as orbifolds.

Every point in a connected, noneffective orbifold O has nontrivial isotropy and hence is singular. We refer to points that correspond to nonsingular points in the associated effective orbifold O_{eff} as *effectively nonsingular* and points that correspond to singular points in the associated effective orbifold as *effectively singular*. An orbifold given by a smooth manifold equipped with the trivial action of a finite group will be referred to as *effectively smooth*.

2.1. **Γ-Sectors of an orbifold.** We recall the following.

Definition 2.2 ([14])**.** Let Γ be a finitely generated discrete group and let O be presented as a quotient orbifold $G \ltimes M$ as above. Let (φ) denote the G-conjugacy class of a homomorphism $\varphi\colon \Gamma \to G$. The *orbifold of* Γ*-sectors* $\widetilde{O}_\Gamma$ of O is the disjoint union of orbifolds presented by

$$\bigsqcup_{(\varphi)\in \mathrm{HOM}(\Gamma,G)/G} C_G(\varphi) \ltimes M^{\langle\varphi\rangle}$$

where $M^{\langle\varphi\rangle}$ denotes the collection of points fixed by each element of the image of φ in G and $C_G(\varphi)$ denotes the centralizer of the image of φ. For a given $\varphi \in \mathrm{HOM}(\Gamma, G)$, we refer to each connected component of the orbifold presented by $C_G(\varphi) \ltimes M^{\langle\varphi\rangle}$ as a Γ-*sector* of O. We let $m_{(\varphi)}$ denote the number of connected components of $C_G(\varphi) \ltimes M^{\langle\varphi\rangle}$ and let $\widetilde{O}^i_{(\varphi)}$ for $i = 1, \ldots, m_{(\varphi)}$ denote the corresponding connected orbifolds. If $C_G(\varphi) \ltimes M^{\langle\varphi\rangle}$ is connected, we denote the corresponding sector simply $\widetilde{O}_{(\varphi)}$. The sector corresponding to the trivial homomorphism $\Gamma \to G$ is diffeomorphic to O and is referred to as the *nontwisted sector* (or *nontwisted sectors* if O is not connected); other sectors are referred to as *twisted sectors*.

If $\Gamma = \mathbb{Z}$, since any homomorphism is completely determined by its value on a generator of $\mathbb{Z}$, there is a bijective correspondence between $\mathrm{HOM}(\mathbb{Z}, G)/G$ and the conjugacy classes of G. In this case, in order to simplify notation, we identify the conjugacy class of a homomorphism $\mathbb{Z} \to G$ with the conjugacy class of the image of a fixed generator of $\mathbb{Z}$. Similarly, $\mathrm{HOM}(\mathbb{Z}^\ell, G)/G$ corresponds to the orbits of commuting ℓ-tuples $(g_1, \ldots, g_\ell) \in G^\ell$ under the action of G by simultaneous conjugation, while $\mathrm{HOM}(\mathbb{F}_\ell, G)/G$, where $\mathbb{F}_\ell$ denotes the free group with ℓ generators, corresponds to the orbits of (not necessarily commuting) ℓ-tuples under simultaneous conjugation.

It is easy to see that each $C_G(\varphi)$ acts with finite isotropy groups on $M^{\langle\varphi\rangle}$ so that $C_G(\varphi) \ltimes M^{\langle\varphi\rangle}$ does indeed present an orbifold. If $\psi = h\varphi h^{-1}$ for some $h \in G$, then left-translation by h induces a diffeomorphism between $M^{\langle\varphi\rangle}$ and $M^{\langle\psi\rangle}$, and conjugation by h intertwines the respective actions of $C_G(\varphi)$ and $C_G(\psi)$, so that the orbifold $C_G(\varphi) \ltimes M^{\langle\varphi\rangle}$ does not depend on the choice of representative φ of (φ). Note that $M^{\langle\varphi\rangle}$ is empty unless the image of φ is contained in the isotropy group of at least one point $x \in M$. One implication is that if O is a smooth manifold with no group action, then $\widetilde{O}_\Gamma = O$ for each Γ. More generally, if $\Gamma = \mathbb{Z}^\ell$ or $\Gamma = \mathbb{F}_\ell$ for $\ell \geq 1$, then $\widetilde{O}_\Gamma = O$ if and only if O is a manifold.

Suppose O is a noneffective orbifold presented by $G \ltimes M$ so that O_{eff} is presented by $G/K \ltimes M$ where K is the finite, normal subgroup of G acting trivially. Let $\rho\colon G \to G/K$ denote the quotient homomorphism. Then for each Γ, there is a surjective map from $\mathrm{HOM}(\Gamma, G)/G$ to $\mathrm{HOM}(\Gamma, G/K)/(G/K)$ given by sending $\varphi \in \mathrm{HOM}(\Gamma, G)$ to $\rho \circ \varphi \in \mathrm{HOM}(\Gamma, G/K)$. It is easy to see that if $\varphi \in \mathrm{HOM}(\Gamma, G)$, then $M^{\langle\varphi\rangle} = M^{\langle\rho\circ\varphi\rangle}$; however the actions of $C_{G/K}(\rho \circ \varphi)$ on $M^{\langle\rho\circ\varphi\rangle}$ and $C_G(\varphi)$ on $M^{\langle\varphi\rangle}$ may differ. Moreover, there are generally more Γ-sectors of O than O_{eff}. In particular, if O is a connected n-dimensional orbifold, only the nontwisted sector of O_{eff} is n-dimensional, while each sector of O corresponding to a $\varphi \in \mathrm{HOM}(\Gamma, K)$ is n-dimensional.

Example 2.3. Let $\mathbb{S}^2$ denote the standard unit sphere, and let $D_6 = \langle a, b : a^3 = b^2 = (ab)^2 = 1\rangle$ denote the dihedral group of order 6. Define a D_6-action on $\mathbb{S}^2$ where $ax = x$ for each $x \in \mathbb{S}^2$, and b acts as a rotation through π about a fixed axis. Then $D_6 \ltimes \mathbb{S}^2$ presents a noneffective orbifold O with $K = \langle a\rangle$ acting trivially. The effectivization O_{eff} is presented by $\langle b\rangle \ltimes \mathbb{S}^2$. The $\mathbb{Z}$-sectors of O_{eff} consist of $\widetilde{(O_{\mathit{eff}})}_{(1)}$, isometric to O_{eff}, and $\widetilde{(O_{\mathit{eff}})}_{(b)}$, a pair of points with trivial $\mathbb{Z}_2$-action. However, the $\mathbb{Z}$-sectors of O consist of $\widetilde{O}_{(1)}$, isometric to O, $\widetilde{O}_{(b)}$, a pair of points with trivial $\mathbb{Z}_2$-action, and $\widetilde{O}_{(a)}$, the standard unit sphere with trivial $\mathbb{Z}_3$-action.

More generally, if O is presented by a Lie groupoid $\mathcal{G}$, then the orbifold of Γ-sectors $\widetilde{O}_\Gamma$ can be constructed as follows. Let $\mathcal{S}_\Gamma^{\mathcal{G}}$ denote the collection of groupoid homomorphisms $\mathrm{HOM}(\Gamma, \mathcal{G})$ treating Γ as a groupoid with a single object. Then $\mathcal{S}_\Gamma^{\mathcal{G}}$ inherits from $\mathcal{G}$ the structure of a union of smooth manifolds (with connected components of different dimensions) as well as a natural $\mathcal{G}$-action. We let $\mathcal{G}^\Gamma$ denote the translation groupoid of $\mathcal{G} \ltimes \mathcal{S}_\Gamma^{\mathcal{G}}$, and then $\mathcal{G}^\Gamma$ is a presentation of $\widetilde{O}_\Gamma$. See [15] for the details of this construction. Note that if $\mathcal{G} = G \ltimes M$ is a translation groupoid, then a groupoid homomorphism $\varphi_x\colon \Gamma \to \mathcal{G}$ corresponds to a choice of $x \in M$ and a group homomorphism $\varphi\colon \Gamma \to G_x \leq G$. In particular, the Γ-sector associated to φ_x using the groupoid definition is a connected orbifold; it corresponds to the connected component of $\widetilde{O}_{(\varphi)}$ containing the orbit of x. If $x \in M$ with isotropy group G_x and $\varphi\colon \Gamma \to G_x \leq G$ is a homomorphism with image contained in G_x, then a linear orbifold chart $\{V_x, G_x, \pi_x\}$ for O at the orbit Gx induces a linear orbifold chart $\left\{V_x^{\langle\varphi\rangle}, C_{G_x}(\varphi), \pi_x^\varphi\right\}$ for $\widetilde{O}_{(\varphi)}$ at the point $C_G(\varphi)x$.

Proposition 2.4 ([14] and [15])**.** *Let Γ be a finitely generated discrete group. A diffeomorphism of orbifolds $O \to O'$ induces a diffeomorphism $\widetilde{O}_\Gamma \to \widetilde{O'}_\Gamma$ for each finitely generated group Γ. If O is compact, then $\widetilde{O}_\Gamma$ is compact, and in particular consists of a finite number of connected components.*

Hence, $\widetilde{O}_\Gamma$ does not depend on the presentation of O. The construction of $\widetilde{O}_\Gamma$ can be though of as an "unraveling" of the singular strata of O into disjoint orbifolds. The choice of Γ corresponds roughly with the depth and type of singular strata that are unraveled. For instance, if $\Gamma = \mathbb{Z}$, then $\widetilde{O}_\Gamma$ is the inertia orbifold, consisting of orbifolds that arise as fixed-point subsets of cyclic groups in O. If $\Gamma = \mathbb{F}_\ell$ is the free group with ℓ generators, then $\widetilde{O}_\Gamma$ corresponds to the ℓ-multisectors; see [1]. Note that the natural projection $\widetilde{O}_\Gamma \to O$ induced by the inclusion of each $M^{\langle\varphi\rangle} \to M$ is not usually injective, even when restricted to a single sector.

Given a metric g on O, a local chart $\{V_x, G_x, \pi_x\}$, and a homomorphism $\varphi\colon \Gamma \to G_x$, the metric g_x on V_x restricts to a $C_{G_x}(\varphi_x)$-invariant metric on $V_x^{\langle\varphi\rangle}$, inducing a metric on the associated Γ-sector. If O is presented by $G \ltimes M$, a choice of corresponding G-invariant metric on the smooth manifold M restricts to each $M^{\langle\varphi\rangle}$ as a

$C_G(\varphi)$-invariant metric, inducing an orbifold metric on each Γ-sector as a quotient orbifold. Using the slice theorem (see e.g. [12, Theorem 2.3.3]), it is easy to see that the restriction of the $C_G(\varphi)$-invariant metric on $M^{\langle\varphi\rangle}$ to a chart $V_x^{\langle\varphi\rangle}$ coincides with the restriction of the local metric g_x to $V_x^{\langle\varphi\rangle}$ so that the metric induced on $\widetilde{O}_\Gamma$ depends only on the metric g and not on the presentation of O. Similarly, given an isometry $O \to O'$ between quotient orbifolds, the induced diffeomorphism $\widetilde{O}_\Gamma \to \widetilde{O'}_\Gamma$ preserves the corresponding local metrics and hence is itself an isometry of the corresponding Γ-sectors.

2.2. **Γ-Spectrum of an orbifold.** For a Riemannian orbifold O, we say that a function is smooth if at every point it can be lifted to a smooth function on a local manifold cover above the point; equivalently, a smooth function is an invariant smooth function on a presentation of O. We denote the space of all smooth functions on O by $C^\infty(O)$. Since the Laplacian Δ acting on smooth function on a manifold commutes with isometries, there is a well-defined action of the Laplacian on $C^\infty(O)$, computed by taking the Laplacian of lifts of smooth functions to local manifold covers. For any compact, connected Riemannian orbifold O, the eigenvalue spectrum of the Laplacian, denoted $\mathrm{Spec}(O)$ is a discrete sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty$, with each eigenvalue appearing with finite multiplicity; see [4], [9]. We say that two orbifolds are *isospectral* if they have the same Laplace spectrum.

Let Γ be a finitely generated discrete group and let O be a compact orbifold presented by $G \ltimes M$ where G is a compact Lie group. Then

$$C^\infty(\widetilde{O}_\Gamma) = \bigoplus_{(\varphi)\in \mathrm{HOM}(\Gamma,G)/G} \bigoplus_{i=1}^{m_{(\varphi)}} C^\infty(\widetilde{O}^i_{(\varphi)}).$$

We let $\Delta^i_{(\varphi)}$ denote the corresponding Laplace operator for each Γ-sector $\widetilde{O}^i_{(\varphi)}$ and set

$$\Delta_\Gamma = \bigoplus_{(\varphi)\in \mathrm{HOM}(\Gamma,G)/G} \bigoplus_{i=1}^{m_{(\varphi)}} \Delta^i_{(\varphi)},$$

so that $\Delta_\Gamma\colon C^\infty(\widetilde{O}_\Gamma) \to C^\infty(\widetilde{O}_\Gamma)$.

Definition 2.5. Let O be a compact Riemannian orbifold and Γ a finitely generated discrete group. The Γ-*spectrum of* O is the spectrum of Δ_Γ acting on $C^\infty(\widetilde{O}_\Gamma)$. In other words

$$\mathrm{Spec}_\Gamma(O) = \bigcup_{(\varphi)\in \mathrm{HOM}(\Gamma,G)/G} \bigcup_{i=1}^{m_{(\varphi)}} \mathrm{Spec}(\widetilde{O}^i_{(\varphi)}).$$

Two compact Riemannian orbifolds O and O' are Γ-*isospectral* if $\mathrm{Spec}_\Gamma(O) = \mathrm{Spec}_\Gamma(O')$.

Remark 2.6. If O is not presented by the quotient of a manifold M by a group G, then the Γ-spectrum of O can be defined identically by applying Spec to the Γ-sectors of O defined using a groupoid presentation of O. By Proposition 2.4, this definition coincides with that given in Definition 2.5 when O admits a presentation as a quotient.

Hence, the Γ-spectrum of O is the usual spectrum of the Γ-sectors of O. If O is a connected Riemannian manifold, then the only Γ-sector of O is the nontwisted sector isometric to O so the Γ-spectrum coincides with the usual notion of the Laplace spectrum. Similarly, if Γ is the trivial group, then $\mathrm{Spec}_\Gamma(O) = \mathrm{Spec}(O)$. Note that the multiplicity of 0 in the Γ-spectrum of O corresponds to the number of Γ-sectors of O.

The usual Laplace spectrum of an orbifold O coincides with the Laplace spectrum of the associated effective orbifold O_{eff}. This is not the case for the Γ-spectrum, as there are generally more Γ-sectors of O than O_{eff}; see Section 2.1 and in particular Example 2.3. In addition, we consider the following.

Example 2.7. Let M be any connected Riemannian manifold, and let O be presented by $\mathbb{Z}_2 \ltimes M$ where the $\mathbb{Z}_2$-action is trivial. Then $\widetilde{M}_\Gamma \cong M$ for any Γ, while $\widetilde{O}_\mathbb{Z} \cong O \sqcup O$. It follows that $\mathrm{Spec}_\mathbb{Z}(M) = \mathrm{Spec}(M)$ while for $\mathrm{Spec}_\mathbb{Z}(O)$, the multiplicity of each eigenvalue from $\mathrm{Spec}(M)$ is doubled. Hence, M and O are isospectral and effectively isometric, though they are not $\mathbb{Z}$-isospectral.

More generally, if a connected orbifold O is presented by $G \ltimes M$ for any finite group G acting trivially, then the number of connected components of $\widetilde{O}_\mathbb{Z}$ coincides with the number of conjugacy classes in G. Each connected component of $\widetilde{O}_\mathbb{Z}$ is effectively isometric to O, though the group acting trivially may vary so that they need not be diffeomorphic.

Example 2.8. Let M be any Riemannian manifold, let O_1 denote the quotient of M by a trivial $\mathbb{Z}_3$-action, and let O_2 denote the quotient of M by a trivial D_6-action where $D_6 = \langle a, b \mid a^3 = b^2 = (ab)^2 = 1\rangle$ is the dihedral group with 6 elements. Then O_1 and O_2 are isospectral and effectively isometric. Moreover, $\widetilde{(O_1)}_\mathbb{Z}$ is isometric to $O_1 \sqcup O_1 \sqcup O_1$, and since D_6 has 3 conjugacy classes (1), (a), and (b), $\widetilde{(O_2)}_\mathbb{Z} \cong D_6 \ltimes M \sqcup \mathbb{Z}_3 \ltimes M \sqcup \mathbb{Z}_2 \ltimes M$ with each group action trivial. Therefore, O_1 and O_2 are also $\mathbb{Z}$-isospectral. However, by counting the conjugacy classes of homomorphisms $\mathbb{Z}^2 \to \mathbb{Z}_3$ and $\mathbb{Z}^2 \to D_6$, it is easy to see that $\widetilde{(O_1)}_{\mathbb{Z}^2}$ has 9 identical connected components while $\widetilde{(O_2)}_{\mathbb{Z}^2}$ has 8 connected components, so that O_1 and O_2 are not $\mathbb{Z}^2$-isospectral. Similarly, $\widetilde{(O_1)}_{\mathbb{F}_2}$ has 9 connected components while $\widetilde{(O_2)}_{\mathbb{F}_2}$ has 12, so that O_1 and O_2 are not $\mathbb{F}_2$-isospectral.

The above examples illustrate that consideration of noneffective orbifolds yields trivial examples of orbifolds that, for instance, are isospectral but not $\mathbb{Z}$-isospectral or are isospectral and $\mathbb{Z}$-isospectral, but not $\mathbb{Z}^2$-isospectral. In addition, consider the following.

Example 2.9. Let $O_1 = G_1 \ltimes M_1$ and $O_2 = G_2 \ltimes M_2$ be any pair of effective isospectral, non-isometric orbifolds (see examples in Section 3), and let p be a prime that does not divide the order of the isotropy group of any point in O_1 or O_2. Then since every homomorphism $\mathbb{Z}_p \to G_i$, $i = 1, 2$ either has empty fixed point set or is trivial, it is easy to see that $\widetilde{(O_1)}_{\mathbb{Z}_p} = O_1$ and $\widetilde{(O_2)}_{\mathbb{Z}_p} = O_2$. Therefore, O_1 and O_2 are also $\mathbb{Z}_p$-isospectral.

Hence, many questions about the Γ-spectrum of a general orbifold have trivial answers that involve algebraic trickery using trivial group actions or a choice of Γ that leads to no nontwisted sectors. We consider the Γ-spectrum to be of most interest when applied to effective Riemannian orbifolds and choices of Γ that yield nontrivial sectors.

2.3. **Elementary Γ-spectral invariants.** Let O be a compact, connected, effective Riemannian orbifold presented as a quotient orbifold by $G \ltimes M$, and let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty$ denote the spectrum of O. The *heat trace* of O is defined to be $\sum_{j=1}^{\infty} e^{-\lambda_j t}$; see [9] and [7]. By [9, Theorem 4.8], the heat trace as $t \to 0^+$ admits an asymptotic expansion of the form

$$(4\pi t)^{-\dim(O)/2} \sum_{j=0}^{\infty} c_j t^{j/2} \tag{2.1}$$

where $c_0 = \mathrm{vol}(O)$ is the Riemannian volume of O. In particular, the volume and dimension of O are determined by the spectrum; see also [13, Theorem 3.2]. It will be useful for us to recall that the asymptotic expansion of the heat trace in Equation 2.1 can also be expressed as follows. Let $\mathcal{S}(O)$ denote the strata of the singular set of O with respect to its Whitney stratification by orbit types. Then the asymptotic expansion of the heat trace can be decomposed into the contributions of the strata as

$$(4\pi t)^{-\dim(O)/2} \sum_{k=0}^{\infty} a_k t^k + \sum_{N \in \mathcal{S}(O)} (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} b_{k,N} t^k \tag{2.2}$$

where the a_k are the usual heat invariants as in the case of manifolds, and $b_{0,S} \neq 0$ for each S. See [7], [18, Lemma 3.3], and [9, Definition 4.7 and Theorem 4.8].

In the case that O is not effective, the heat trace of O coincides with that of O_{eff} since $\mathrm{Spec}(O) = \mathrm{Spec}(O_{\mathit{eff}})$. However, if K denotes the isotropy group of an effectively nonsingular point, then $\mathrm{vol}(O) = \mathrm{vol}(O_{\mathit{eff}})/|K|$. To see this, note that if $\{V_x, G_x, \pi_x\}$ is an orbifold chart, a differential form ω on $\pi_x(V_x) \subseteq O$ can be defined locally as a G_x-invariant differential form on V_x, and the integral of ω on $\pi_x(V_x)$ is defined to be

$$\int_{\pi_x(V_x)} \omega := \frac{1}{|G_x|} \int_{V_x} \pi_x^* \omega,$$

see [1, p. 34]. If $p = \pi_x(x)$ is an effectively nonsingular point in O, then $G_x \cong K$. For an arbitrary point, if G_x^{eff} denotes the isotropy group of the point corresponding to $\pi_x(x)$ in O_{eff}, then $|G_x^{\mathit{eff}}| = |G_x|/|K|$ so that this integral differs from the integral of the corresponding ω on O_{eff} by a factor of $|K|$. Hence, if O is not effective, then the volume can be determined from the spectrum along with the order of the group K acting trivially, but cannot be determined from the spectrum alone.

For a finitely generated discrete group Γ, the Γ-*heat trace of* O is defined to be the heat trace of $\widetilde{O}_\Gamma$. It is evidently given by the sum of the heat traces of the sectors of $\widetilde{O}_\Gamma$. Specifically, for each conjugacy class $(\varphi) \in \mathrm{HOM}(\Gamma, G)/G$ and each $i = 1, \ldots, m_{(\varphi)}$, let $0 = \lambda_0((\varphi), i) < \lambda_1((\varphi), i) \leq \lambda_2((\varphi), i) \leq \cdots \uparrow \infty$ denote the spectrum of the closed orbifold $\widetilde{O}_{(\varphi)}^i$, the corresponding connected component of $C_G(\varphi) \ltimes M^{\langle \varphi \rangle}$. Let $H_{(\varphi),i}(t) = \sum_{j=0}^{\infty} e^{-\lambda_j((\varphi),i)t}$ denote the corresponding heat trace of $\widetilde{O}^i_{(\varphi)}$. Then the Γ-heat trace of O is given by

$$\sum_{(\varphi)\in\mathrm{HOM}(\Gamma,G)/G} \quad \sum_{i=1}^{m_{(\varphi)}} H_{(\varphi),i}(t).$$

Then the Γ-heat trace is asymptotic as $t \to 0^+$ to

$$\sum_{(\varphi)\in\mathrm{HOM}(\Gamma,G)/G} \quad \sum_{i=1}^{m_{(\varphi)}} (4\pi t)^{-\dim(\widetilde{O}^i_{(\varphi)})/2} \sum_{j=0}^{\infty} c_j((\varphi),i) t^{j/2},$$

where the $c_j((\varphi),i)$ are the coefficients of the asymptotic expansion of the heat trace of $\widetilde{O}^i_{(\varphi)}$.

If O is effective, then the dimension of the nontwisted sector $\widetilde{O}_{(1)} \cong O$ is strictly larger than the dimension of each twisted sector. Therefore, the lowest-degree term in the asymptotic expansion of the heat trace is $(4\pi t)^{-\dim(O)/2}\mathrm{vol}(O)$, with no contributions from the twisted sectors. In particular, the lowest degree term of the asymptotic expansions of the Γ-heat trace and ordinary heat trace coincide.

Suppose on the other hand that O is not effective so that a nontrivial finite subgroup $K \trianglelefteq G$ acts trivially on M. Then the sectors $\widetilde{O}_{(\varphi)}$ corresponding to φ with image contained in K have dimension equal to $\dim(O)$, while all other sectors have dimension strictly less than $\dim(O)$. Then as G acts on $\mathrm{HOM}(\Gamma, K)$, it follows that the lowest-degree term in the asymptotic expansion of the heat trace is

$$(4\pi t)^{-\dim(O)/2} \sum_{(\varphi)\in\mathrm{HOM}(\Gamma,K)/G} \mathrm{vol}\left(C_G(\varphi) \ltimes M\right),$$

where the volumes in the sum need not be of connected orbifolds.

These observations yield the following.

Proposition 2.10. *Let O be a compact, connected, effective Riemannian orbifold. Then the volume and dimension of O are determined by the Γ-spectrum for any finitely generated discrete group Γ. If O is not effective, then the dimension of O is determined by the Γ-spectrum for any finitely generated discrete group Γ.*

Note that as in the case of the ordinary spectrum, the asymptotic expansion of the heat trace is a strictly coarser invariant than the spectrum itself. See Section 3.3.3 for an example of orbifolds for which the asymptotic expansions of the Γ-heat traces coincide for every group Γ though the orbifolds are not Γ-isospectral for every Γ.

3. Isospectral, nonisometic orbifolds and their Γ-spectra

In this section, we consider the Γ-spectra of examples of isospectral, nonisometric orbifolds that have been given in the literature. Along with using these examples to illustrate features of the Γ-spectrum, we will see that in many examples, the $\mathbb{Z}$-spectrum is able to distinguish between isospectral pairs; this is the case for the examples recalled in Sections 3.1 through 3.3. We will also indicate known examples of nonisometric orbifolds with nontrivial singular set that are Γ-isospectral for every choice of Γ in Section 3.4. Throughout this section, we restrict our attention to pairs of orbifolds with nontrivial singular sets because if O and O' are isospectral manifolds, then they are automatically Γ-isospectral for every choice of Γ.

3.1. **Examples of Shams-Stanhope-Webb.** In [31], given an odd prime p and an integer $m \geq 1$, Shams, Stanhope, and Webb construct a family $\{O_i : i = 0, \ldots, m\}$ of pairwise isospectral, nonisometric orbifolds. As we will see below, no O_i and O_j with $i \neq j$ are $\mathbb{Z}$-isospectral. These examples illustrate how one can conclude that orbifolds are not $\mathbb{Z}$-isospectral without determining the $\mathbb{Z}$-sectors explicitly.

The orbifolds O_i are given by quotients of the standard unit sphere $\mathbb{S}^{p^{3m}-1}$ in $\mathbb{R}^{p^{3m}}$ by subgroups of the permutation group $S_{p^{3m}}$ acting on a basis for $\mathbb{R}^{p^{3m}}$. Specifically, let H denote the mod-p Heisenberg group and let $E = (\mathbb{Z}_p)^3$. Then O_i is given by $H_i \ltimes \mathbb{S}^{p^{3m}-1}$ where $H_i = H^i \times E^{m-i}$, realized as a subgroup of $S_{p^{3m}}$.

To see that no pair O_i and O_j are $\mathbb{Z}$-isospectral for $i \neq j$, we first consider the case $m = 1$, i.e. the two orbifolds $O_0 = E \ltimes \mathbb{S}^{p^3-1}$ and $O_1 = H \ltimes \mathbb{S}^{p^3-1}$. To compute the fixed-point sets, note that each nontrivial element a of E has order p so that the left action of a partitions E into p^2 orbits of size p. Given a standard basis vector e_i of $\mathbb{R}^{p^3}$, let $e_i^a = \sum_{k=0}^{p-1} a^k e_i$ denote the average of e_i over the action of $\langle a \rangle$. Note that $e_i^a = e_j^a$ if and only if e_i and e_j are in the same orbit, and hence there are p^2 distinct e_i^a corresponding to the p^2 $\langle a \rangle$-orbits in E. A vector in $\mathbb{R}^{p^3}$ is fixed by a if and only if it is a linear combination of the e_i^a, so that $(\mathbb{R}^{p^3})^a$ is a subspace of dimension p^2. Then $(\mathbb{S}^{p^3-1})^a$ is (p^2-1)-dimensional, and in particular is a subsphere of $\mathbb{S}^{p^3-1}$ of positive dimension, hence connected. As E and H are almost conjugate, the same holds for true for the nontrivial elements of H.

Since E is abelian, the $\mathbb{Z}$-sectors $\widetilde{(O_0)}_{\mathbb{Z}}$ consist of the nontwisted sector O_0 as well as $p^3 - 1$ twisted sectors of the form $E \ltimes \mathbb{S}^{p^2-1}$. Thus, $\widetilde{(O_0)}_{\mathbb{Z}}$ has p^3 connected components. On the other hand, since H is not abelian, H contains strictly fewer than p^3 conjugacy classes and therefore $\widetilde{(O_1)}_{\mathbb{Z}}$ has strictly fewer than p^3 connected components, each of which can be represented by a quotient of $\mathbb{S}^{p^2-1}$ by a subgroup of H. We conclude that O_0 and O_1 are not $\mathbb{Z}$-isospectral since the multiplicities of 0 in their $\mathbb{Z}$-spectra do not coincide.

For general m, note that the bijection from $\mathrm{HOM}(\mathbb{Z}, A \times B)$ to $\mathrm{HOM}(\mathbb{Z}, A) \times \mathrm{HOM}(\mathbb{Z}, B)$ given by $\varphi \mapsto (\pi_A \circ \varphi, \pi_B \circ \varphi)$, where π_A and π_B denote the respective projection homomorphisms, is equivariant with respect to conjugation by $A \times B$ and hence induces a bijection between $\mathrm{HOM}(\mathbb{Z}, A \times B)/(A \times B)$ and $\mathrm{HOM}(\mathbb{Z}, A)/A \times \mathrm{HOM}(\mathbb{Z}, B)/B$. Hence, the above argument demonstrates that for $i < j$, the $\mathbb{Z}$-sectors of $H_i \ltimes \mathbb{S}^{p^{3m}-1}$ have strictly fewer connected components than the $\mathbb{Z}$-sectors of $H_j \ltimes \mathbb{S}^{p^{3m}-1}$. Thus, we conclude that no pair of these orbifolds is $\mathbb{Z}$-isospectral.

3.2. **Homogeneous space examples of Rossetti-Schueth-Weilandt.** In [28], Rossetti, Schueth, and Weilandt describe several pairs of isospectral, nonisometric orbifolds, demonstrating in particular that isospectral orbifolds need not have the same maximal isotropy order. The first three examples that they give (Examples 2.7–9) are biquotients of $SO(6)$. In each of these examples, the resulting isospectral orbifold pairs are not $\mathbb{Z}^\ell$- or $\mathbb{F}_\ell$-isospectral for any $\ell \geq 1$. We will describe this computation for their Examples 2.7 and 2.9 explicitly in Sections 3.2.1 and 3.2.2 to illustrate the computation of the Γ-sectors and Γ-spectrum. Note that the computation for their Example 2.8 is similar to that carried out for Example 2.7 below; the resulting twisted sectors of the two orbifolds have a common Riemannian cover and can be seen to be not isospectral by an application of [18, Proposition 3.4(ii)].

In order to describe these examples, let $a_{i_1 i_2 \cdots i_m}$ to denote a square diagonal 6×6 matrix with -1 in the $i_1, i_2, \dots, i_m$ positions and 1 everywhere else. We recall from [28, Example 2.4] the groups

$$K_1 = \{I, -I, a_{12}, a_{13}, a_{23}, a_{1456}, a_{2456}, a_{3456}\} \tag{3.1}$$

and

$$K_2 = \{I, -I, a_{12}, a_{34}, a_{56}, a_{1234}, a_{1256}, a_{3456}\}. \tag{3.2}$$

The orbifold pairs in Examples 2.7-9 in [28] are of the form $O_1 = K_1 \ltimes (SO(6)/H)$ and $O_2 = K_2 \ltimes (SO(6)/H)$ where H is a choice of subgroup in $SO(6)$. They are isospectral by Sunada's theorem. (See Section 4 below for a discussion of Sunada's theorem.)

3.2.1. *$H \cong SO(3)$, [28, Example 2.7].* In this case, $H \cong SO(3)$ is chosen to be the subgroup of matrices in 3×3 blocks of the form

$$\begin{bmatrix} A & 0 \\ 0 & I_3 \end{bmatrix}$$

with $A \in SO(3)$, so that $M = SO(6)/H$ is the Steifel manifold $V_{6,3}$ of orthonormal 3-frames in $\mathbb{R}^6$. In particular, for $b \in SO(6)$, the coset bH corresponds to the frame given by the last three columns of b. With a choice of biinvariant metric on $SO(6)$, the orbifolds $O_1 = K_1 \ltimes M$ and $O_2 = K_2 \ltimes M$ are isospectral with different maximal isotropy orders.

To compute the $\mathbb{Z}$-sectors of O_1 and O_2, note that an element $bH \in SO(6)/H$ is fixed by $a_{i_1 i_2 \cdots i_m}$ (acting on the left) if and only if the last three columns of b have the zero rows in positions $i_1, i_2, \dots, i_m$. It is easy to see that

$$M^{a_{1456}} = M^{a_{2456}} = M^{a_{3456}} = M^{a_{1234}} = M^{a_{1256}} = M^{-I} = \emptyset,$$

because if bH were fixed by any of these elements of K_1 or K_2, then the last three columns of b would be a linearly independent set of three vectors in a subspace of dimension zero or two. Thus the $\mathbb{Z}$-sectors of O_1 and O_2 both consist of four connected components.

In the case of O_1, besides the nontwisted sector, the three $\mathbb{Z}$-sectors

$$\widetilde{(O_1)}_{(a_{12})} \cong \widetilde{(O_1)}_{(a_{13})} \cong \widetilde{(O_1)}_{(a_{23})}$$

are isometric; we describe $\widetilde{(O_1)}_{(a_{12})}$ in detail. The fixed point set $M^{a_{12}}$ consists of cosets bH corresponding to elements $b \in SO(6)$ of the form

$$\begin{bmatrix} *_{2\times 3} & 0_{2\times 3} \\ *_{4\times 3} & v_{4\times 3} \end{bmatrix}.$$

Here, $v_{4\times 3}$ is an orthonormal 3-frame in $\mathbb{R}^4$ that depends only on the coset bH, while $*$ indicates entries of b that depend on the choice of representative from bH. Since K_1 is abelian, the centralizer of any element is the entire group, so $\widetilde{(O_1)}_{(a_{12})} \cong K_1 \ltimes M^{a_{12}}$.

In order to understand the action of K_1 on $M^{a_{12}}$, we first note that $K_1 \cong (\mathbb{Z}_2)^3 \cong \langle a_{12}\rangle \oplus \langle a_{13}\rangle \oplus \langle a_{1456}\rangle$. Thus the action of K_1 corresponds to a trivial $\mathbb{Z}_2$-action generated by a_{12} as well as a nontrivial $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action generated by a_{13} and a_{1456} on the rows indexed 3456 of $v_{4\times 3}$. The action of a_{13} fixes the set of cosets for which the first row of $v_{4\times 3}$ vanishes, and therefore the fixed point set of a_{13} is isometric to $SO(3)$. All other elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ have no fixed points because, as above, it is impossible to have three linearly independent vectors in a subspace of dimension less than three.

The remaining twisted sectors $\widetilde{(O_1)}_{(a_{13})}$ and $\widetilde{(O_1)}_{(a_{23})}$ are identical up to permuting rows.

In the case of O_2, the three twisted $\mathbb{Z}$-sectors are

$$\widetilde{(O_2)}_{(a_{12})} \cong \widetilde{(O_2)}_{(a_{34})} \cong \widetilde{(O_2)}_{(a_{56})}$$

and are again isometric simply by permuting rows, so we focus our attention on $\widetilde{(O_2)}_{(a_{12})}$. The fixed point set $M^{a_{12}}$ is as in the case of O_1, and again, since K_2 is abelian, $\widetilde{(O_2)}_{(a_{12})} \cong K_2 \ltimes M^{a_{12}}$. In this case, $K_2 \cong (\mathbb{Z}_2)^3 \cong \langle a_{12}\rangle \oplus \langle a_{34}\rangle \oplus \langle a_{56}\rangle$ so the action of K_2 corresponds to a trivial $\mathbb{Z}_2$-action generated by a_{12} as well as a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action generated by a_{34} and a_{56} on the rows indexed 3456 of $v_{4\times 3}$. As above, since there cannot be three linearly independent vectors in a subspace of dimension less than three, the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action is free. We conclude therefore that $\widetilde{(O_2)}_{(a_{12})}$ (and thus $\widetilde{(O_2)}_{(a_{34})}$ and $\widetilde{(O_2)}_{(a_{56})}$) is a smooth manifold.

To see that O_1 and O_2 are not $\mathbb{Z}$-isospectral, note that the $\mathbb{Z}$-spectrum of each O_i is the union

$$\mathrm{Spec}(O_i) \cup 3\mathrm{Spec}\left(\widetilde{(O_i)}_{(a_{12})}\right),$$

where the 3 indicates that the multiplicity of each element of the spectrum is multiplied by 3. Since O_1 and O_2 are isospectral, it is sufficient to show that $\widetilde{(O_1)}_{(a_{12})}$ and $\widetilde{(O_2)}_{(a_{12})}$ are not isospectral. The effective orbifold associated to $\widetilde{(O_1)}_{(a_{12})}$ is a 6-dimensional orbifold with 3-dimensional singular set, while the effective orbifold associated to $\widetilde{(O_2)}_{(a_{12})}$ is a smooth 6-dimensional manifold. Hence, it follows from [9, Theorem 5.1] that they are not isospectral.

It is also of interest to consider the Γ-sectors of the orbifolds O_1 and O_2 for other free groups Γ. Since K_1 and K_2 are abelian, the $\mathbb{Z}^2$-sectors and the $\mathbb{F}_2$-sectors coincide; see [11]. The fact that the $\mathbb{Z}^2$-sectors coincide with the $\mathbb{Z}$-sectors of the $\mathbb{Z}$-sectors computed above (see [16, Theorem 3.1]) makes it straightforward to compute the $\mathbb{Z}^2$-sectors of O_1 and O_2.

For O_1, from the nontwisted $\mathbb{Z}$-sector O_1, a computation identical to the one above gives four $\mathbb{Z}^2$-sectors: one copy of O_1 and three sectors that are isometric to $\widetilde{(O_1)}_{(a_{12})} \cong K_1 \ltimes M^{a_{12}}$. From each of the three twisted $\mathbb{Z}$-sectors $\widetilde{(O_1)}_{(a_{12})}$ we get two copies of $\widetilde{(O_1)}_{(a_{12})}$ (correponding to homomorphisms $\mathbb{Z} \to K_1$ with image I and $\langle a_{12}\rangle$ respectively) and two copies of $K_1 \ltimes SO(3)$ (corresponding to homomorphisms $\mathbb{Z} \to K_1$ with images $\langle a_{13}\rangle$ and $\langle a_{23}\rangle$ respectively). Thus in total, the $\mathbb{Z}^2$-sectors of O_1 are given by

- the nontwisted sector isometric to O_1;
- nine isometric copies of $\widetilde{(O_1)}_{(a_{12})}$; and
- six isometric copies of $K_1 \ltimes SO(3)$.

Similarly, for O_2, from the nontwisted $\mathbb{Z}$-sector O_2 we obtain one copy of O_2 and three sectors that are isometric to $\widetilde{(O_2)}_{(a_{12})} \cong K_2 \ltimes M^{a_{12}}$. From each of the three twisted $\mathbb{Z}$-sectors $\widetilde{(O_2)}_{(a_{12})}$ we get two copies of $\widetilde{(O_2)}_{(a_{12})}$, corresponding to homomorphisms $\mathbb{Z} \to K_2$ with image I and $\langle a_{12}\rangle$ respectively. Thus the $\mathbb{Z}^2$-sectors of O_2 are

- the nontwisted sector isometric to O_2; and
- nine isometric copies of $\widetilde{(O_2)}_{(a_{12})}$.

Since the number of connected components of the $\mathbb{Z}^2$-sectors of O_1 and O_2 respectively do not coincide, the multiplicities of 0 in the $\mathbb{Z}^2$-spectra of O_1 and O_2 do not coincide, thus O_1 and O_2 are not $\mathbb{Z}^2$-isospectral.

As the isotropy group of each point in O_1 and O_2 is abelian and can be generated by two elements, the $\mathbb{Z}^\ell$-sectors of each O_i for $\ell > 2$ will simply yield multiple copies of the $\mathbb{Z}^2$-sectors of O_i. In general, by counting homomorphisms $\mathbb{Z}^\ell \to K_i$ whose images have nontrivial fixed point sets, we conclude that $\widetilde{(O_1)}_{\mathbb{Z}^\ell}$ consists of 4^ℓ connected components while $\widetilde{(O_2)}_{\mathbb{Z}^\ell}$ consists of $3 \cdot 2^\ell - 2$ connected components, so these orbifolds are not $\mathbb{Z}^\ell$-isospectral for any positive ℓ.

3.2.2. $H \cong SO(5)$ [28, Example 2.9]. In this case, $H \cong SO(5)$ is chosen to be the subgroup of matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

with $A \in SO(5)$ so that $M = SO(6)/H$ is isometric to the standard sphere $\mathbb{S}^5$ in $\mathbb{R}^6$. The twisted $\mathbb{Z}$-sectors of the orbifold $O_1 = K_1 \ltimes M$ are given by

$$\widetilde{(O_1)}_{(a_{12})} \cong \widetilde{(O_1)}_{(a_{13})} \cong \widetilde{(O_1)}_{(a_{23})},$$

each isometric to the standard sphere $\mathbb{S}^3$ with trivial $\mathbb{Z}_2$-action generated by a_{12} and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action generated by a_{13} and a_{1456} in coordinates indexed 3456 for $\mathbb{R}^4$, as well as

$$\widetilde{(O_1)}_{(a_{1456})} \cong \widetilde{(O_1)}_{(a_{2456})} \cong \widetilde{(O_1)}_{(a_{3456})},$$

each isometric to $\mathbb{S}^1$ with trivial $\mathbb{Z}_2$-action generated by a_{1456} and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action generated by a_{12} and a_{13} in coordinates indexed 23 for $\mathbb{R}^2$.

Similarly, the twisted $\mathbb{Z}$-sectors of $O_2 = K_2 \ltimes M$ are given by

$$\widetilde{(O_2)}_{(a_{12})} \cong \widetilde{(O_2)}_{(a_{34})} \cong \widetilde{(O_2)}_{(a_{56})},$$

each isometric to the standard sphere $\mathbb{S}^3$ with trivial $\mathbb{Z}_2$-action generated by a_{12} and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action generated by a_{34} and a_{56} in coordinates indexed 3456 for $\mathbb{R}^4$, as well as

$$\widetilde{(O_2)}_{(a_{1234})} \cong \widetilde{(O_2)}_{(a_{1256})} \cong \widetilde{(O_2)}_{(a_{3456})},$$

each isometric to $\mathbb{S}^1$ with trivial $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action generated by a_{12} and a_{34}, and $\mathbb{Z}_2$-action generated by a_{56}.

It is possible to compute the small values of the $\mathbb{Z}$-spectrum directly using the fact that the eigenfunctions of the Laplacian on a standard sphere are given by the restrictions of the homogeneous harmonic polynomials on $\mathbb{R}^n$; see [2]. It follows that the eigenfunctions on an orbifold space form are given by the invariant homogeneous harmonic polynomials; see [28]. By computing bases for the k^{th} eigenspaces of $\mathbb{S}^1$ and $\mathbb{S}^3$ and checking invariance directly, one computes that the first elements of the spectrum of $\widetilde{(O_1)}_{(a_{1456})}$ are 0 and 4, both with a multiplicity of 1. The next eigenvalue of $\widetilde{(O_1)}_{(a_{1456})}$ must be at least 9. The first eigenvalue of $\widetilde{(O_1)}_{(a_{12})}$ is 0 with a multiplicity of 1 and the next eigenvalue is 8.

On the other hand, the first elements of the spectrum of $\widetilde{(O_2)}_{(a_{1234})}$ are 0 with a multiplicity of 1 and 4 with a multiplicity of 2, and the next eigenvalue is at least 9. The first eigenvalue of $\widetilde{(O_2)}_{(a_{12})}$ is 0 with a multiplicity of 1, and the next eigenvalue is 8.

It follows that $\mathbb{Z}$-spectrum of O_1 is given by

$$\operatorname{Spec}(O_1) \cup \{0_6, 4_3, \cdots\}$$

with subscripts indicating multiplicity, while the $\mathbb{Z}$-spectrum of O_2 is given by

$$\operatorname{Spec}(O_2) \cup \{0_6, 4_6, \cdots\}.$$

Since $\operatorname{Spec}(O_1) = \operatorname{Spec}(O_2)$, the multiplicity of 4 cannot coincide in the $\mathbb{Z}$-spectra of O_1 and O_2, and hence they are not $\mathbb{Z}$-isospectral.

Similarly, because $\widetilde{(O_1)}_{(a_{12})}$, $\widetilde{(O_1)}_{(a_{1456})}$, and $\widetilde{(O_2)}_{(a_{12})}$ all have 6 $\mathbb{Z}$-sectors, while $\widetilde{(O_2)}_{(a_{1234})}$ has 4 $\mathbb{Z}$-sectors, it is easy to see that $\widetilde{(O_1)}_{\mathbb{Z}^\ell} = \widetilde{(O_1)}_{\mathbb{F}_\ell}$ has more connected components than $\widetilde{(O_2)}_{\mathbb{Z}^\ell} = \widetilde{(O_2)}_{\mathbb{F}_\ell}$, so that O_1 and O_2 are not $\mathbb{Z}^\ell$- or $\mathbb{F}_\ell$-isospectral for any ℓ.

3.3. **Flat space examples of Rossetti-Schueth-Weilandt.** In addition to the examples given above, Rossetti, Scheuth, and Weilandt also describe pairs of isospectral, nonisometric orbifolds given by quotients of $\mathbb{R}^3$ with its standard, flat metric by pairs of crystallographic groups K_1 and K_2, i.e. groups of isometries of $\mathbb{R}^3$ that act properly discontinuously with compact quotients. In these cases, the resulting orbifolds are shown to be isospectral using either Sunada's theorem or an eigenspace dimension counting formula [28, Theorem 3.1]; see also [21, 22]. In every case, the resulting orbifolds are not $\mathbb{Z}^\ell$- or $\mathbb{F}_\ell$-isospectral for $\ell \geq 1$. This follows from the fact that the collections of twisted sectors are not isospectral, similar to the examples in Section 3.2. However, because the singular sets of these orbifolds are described in detail in [28], the sectors can be computed directly from these descriptions using Remark 3.1 below. We will briefly describe the sectors and consequence for Examples 3.3, 3.5, and 3.7 of [28] in order to illustrate this approach; note that Examples 3.9 and 3.10 can be treated identically. In each case, $O_i = K_i \ltimes \mathbb{R}^3$ for $i = 1, 2$. See [38] for more details.

Remark 3.1. Let O be a quotient orbifold represented by $G \ltimes M$, Γ a finitely generated discrete group, and $\varphi\colon \Gamma \to G$ a homomorphism such that $M^{\langle\varphi\rangle} \neq \emptyset$. Recall that a linear orbifold chart $\{V_x, G_x, \pi_x\}$ for O at the orbit Gx induces a chart $\left\{V_x^{\langle\varphi\rangle}, C_{G_x}(\varphi), \pi_x^\varphi\right\}$ for $\widetilde{O}_{(\varphi)}$ at the point $C_G(\varphi)x$. Hence, the Γ-sectors can be determined locally in terms of an orbifold chart and then patched together, which is often convenient when the singular set and isotropy groups of O are known explicitly. We use this fact when computing sectors in each of the examples in this section.

3.3.1. [28, Example 3.3]. In this example, the singular sets of the orbifolds O_1 and O_2 each consist of a disjoint union of circles. For each $\Gamma = \mathbb{Z}^\ell$ or $\mathbb{F}_\ell$ with $\ell \geq 1$, the Γ-sectors of O_1 and O_2 have a different number of connected components so that O_1 and O_2 are not Γ-isospectral.

The singular set of O_1 consists of three circles $\mathbb{S}^1$ of length 1 with isotropy groups $\mathbb{Z}_4$, $\mathbb{Z}_4$, and $\mathbb{Z}_2$, respectively. A linear orbifold chart for a point contained in a singular circle with isotropy group $\mathbb{Z}_k$ is of the form $\{V_x, G_x, \pi_x\}$ where V_x is diffeomorphic to $\mathbb{R}^3$ and $G_x \cong \mathbb{Z}_k$ acts as rotations about an axis. Hence the corresponding charts for $\widetilde{(O_1)}_{\mathbb{Z}}$ are parameterized by homomorphisms $\varphi\colon \mathbb{Z} \to \mathbb{Z}_k$, where the trivial homomorphism yields a chart for the nontwisted sector and each nontrivial homomorphism yields a chart of the form $\left\{V_x^{\langle\varphi\rangle}, C_{G_x}(\varphi), \pi_x^{\varphi}\right\}$ where $V_x^{\langle\varphi\rangle}$ is a line on which $C_{G_x}(\varphi) \cong \mathbb{Z}_k$ acts trivially. These charts patch together to describe a neighborhood of the circle in the nontwisted sector as well as one circle with trivial $\mathbb{Z}_k$-action for each nontrivial homomorphism $\varphi\colon \mathbb{Z} \to \mathbb{Z}_k$.

It follows that the twisted $\mathbb{Z}$-sectors of O_1 consist of seven copies of $\mathbb{S}^1$ with length 1, six with trivial $\mathbb{Z}_4$-action and one with trivial $\mathbb{Z}_2$-action. The singular set of O_2, on the other hand, consists of four copies of $\mathbb{S}^1$, each with $\mathbb{Z}_2$-isotropy, two of length 2 and two of length 1. Therefore, the twisted $\mathbb{Z}$-sectors of O_2 consist of four copies of $\mathbb{S}^1$ with trivial $\mathbb{Z}_2$-action in pairs of length 2 and 1. As the numbers of connected components do not coincide, O_1 and O_2 are not $\mathbb{Z}$-isospectral. Similarly, for $\ell > 1$, O_1 and O_2 are easily seen to not be $\mathbb{Z}^\ell$- or $\mathbb{F}_\ell$-isospectral by counting numbers of nontrivial homomorphisms from $\mathbb{Z}^\ell$ into respective isotropy groups. Indeed, $\widetilde{(O_1)}_{\mathbb{Z}^\ell} = \widetilde{(O_1)}_{\mathbb{F}_\ell}$ has $2(4^\ell - 1) + 2^\ell$ connected components, each twisted sector a circle of length 1 with $\mathbb{Z}_2$- or $\mathbb{Z}_4$-isotropy, while $\widetilde{(O_2)}_{\mathbb{Z}^\ell} = \widetilde{(O_2)}_{\mathbb{F}_\ell}$ has $4 \cdot 2^\ell - 3$ connected components, each twisted sector a circle of length 1 or 2 with $\mathbb{Z}_2$-isotropy.

3.3.2. [28, Example 3.5]. This example is similar to that treated in Section 3.3.1 above, though the singular set of O_2 is more interesting. We again have that the Γ-sectors of O_1 and O_2 have a different number of connected components when $\Gamma = \mathbb{Z}^\ell$ or $\mathbb{F}_\ell$ with $\ell \geq 1$.

The singular set of O_1 consists of three circles $\mathbb{S}^1$ of length 2 with isotropy groups $\mathbb{Z}_4$, $\mathbb{Z}_4$, and $\mathbb{Z}_2$, respectively. It follows that the twisted $\mathbb{Z}$-sectors of O_1 consist of 7 copies of $\mathbb{S}^1$ with length 2, six with trivial $\mathbb{Z}_4$-action and one with trivial $\mathbb{Z}_2$-action. The orbifold O_2 has singular set given by a trivalent graph with 8 vertices and 12 edges forming the 1-skeleton of a cube, where each vertex has $\mathbb{Z}_2 \times \mathbb{Z}_2$-isotropy and each edge has $\mathbb{Z}_2$-isotropy; the action of each $\mathbb{Z}_2$ on $\mathbb{R}^3$ is given by a rotation through π about a single axis, while the action of each $\mathbb{Z}_2 \times \mathbb{Z}_2$ is given by rotation through π about two orthogonal axes. Because the vertices do not have cyclic isotropy, they do not appear as 0-dimensional $\mathbb{Z}$-sectors. Rather, the twisted $\mathbb{Z}$-sectors are twelve mirrored intervals of length 1 with $\mathbb{Z}_2$-isotropy on the interior and $\mathbb{Z}_2 \times \mathbb{Z}_2$-isotropy on the endpoints. That is, each twisted sector is the quotient of a circle $\mathbb{S}^1$ of length 2 by a $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, where one $\mathbb{Z}_2$-factor acts trivially and the other acts by reflection through a diameter. Since the numbers of connected components do not match, O_1 and O_2 are not $\mathbb{Z}$-isospectral. Similarly, for $\ell > 1$, O_1 and O_2 are easily seen to not be $\mathbb{Z}^\ell$- or $\mathbb{F}_\ell$-isospectral. Indeed, $\widetilde{(O_1)}_{\mathbb{Z}^\ell} = \widetilde{(O_1)}_{\mathbb{F}_\ell}$ has $2(4^\ell - 1) + 2^\ell$ connected components, each twisted sector a circle of length 2 with $\mathbb{Z}_2$- or $\mathbb{Z}_4$-isotropy. On the other hand, $\widetilde{(O_2)}_{\mathbb{Z}^\ell} = \widetilde{(O_2)}_{\mathbb{F}_\ell}$ has $2^{2\ell+3} - 3 \cdot 2^{\ell+2} + 5$ connected components consisting of $8(4^\ell - 3 \cdot 2^\ell + 2)$ points with $\mathbb{Z}_2 \times \mathbb{Z}_2$-isotropy, $12(2^\ell - 1)$ mirrored intervals with $\mathbb{Z}_2$-isotropy on the interior, and the nontwisted sector.

3.3.3. [28, Example 3.7]. The orbifolds in this example are again similar to those in Section 3.3.1 above, and are not Γ-isospectral for any Γ that admits a nontrivial homomorphism to $\mathbb{Z}_2$. However, it is of interest to note that the asymptotic expansions of the Γ-heat kernels of O_1 and O_2 coincide for every group Γ; see Section 2.3.

The orbifold O_1 in this case has a singular set consisting of two copies of $\mathbb{S}^1$ of length $\sqrt{2}$ with $\mathbb{Z}_2$-isotropy. Hence the twisted $\mathbb{Z}$-sectors consist of two copies of $\mathbb{S}^1$ of length $\sqrt{2}$ with trivial $\mathbb{Z}_2$-action. The singular set of O_2 consists of four copies of $\mathbb{S}^1$ of length $1/\sqrt{2}$ with $\mathbb{Z}_2$-isotropy, so that the twisted $\mathbb{Z}$-sectors consist of four copies of $\mathbb{S}^1$ of length $1/\sqrt{2}$ equipped with trivial $\mathbb{Z}_2$-action. Again, the $\mathbb{Z}$-sectors have different numbers of connected components, and hence O_1 and O_2 are not $\mathbb{Z}$-isospectral. Similarly, for any Γ, it is easy to see that $\widetilde{(O_1)}_\Gamma$ consists of the nontwisted sector and $2(|\mathrm{HOM}(\Gamma,\mathbb{Z}_2)|-1)$ circles of length $\sqrt{2}$ with $\mathbb{Z}_2$-isotropy, while $\widetilde{(O_2)}_\Gamma$ consists of the nontwisted sector and $4(|\mathrm{HOM}(\Gamma,\mathbb{Z}_2)|-1)$ circles of length $1/\sqrt{2}$ with $\mathbb{Z}_2$-isotropy. Therefore, O_1 and O_2 are not Γ-isospectral for any Γ that admits a nontrivial homomorphism to $\mathbb{Z}_2$.

To see that that the asymptotic expansions of the Γ-heat kernels of O_1 and O_2 coincide for every Γ, note that as O_1 and O_2 are isospectral, the usual heat kernels of O_1 and O_2 coincide. In addition, the asymptotic expansion of the heat kernel of a 1-dimensional manifold with connected components $M_1,\ldots,M_n$ is given by $(l_1+\cdots+l_n)(4\pi t)^{-\frac{1}{2}}$ where l_i is the length of M_i; see [3, Section 9] or [24, Section 1.2]. Hence, as the twisted Γ-sectors of O_1 and O_2 consist of circles whose lengths sum to $2\sqrt{2}(|\mathrm{HOM}(\Gamma,\mathbb{Z}_2)|-1)$, the contributions of the twisted sectors to the asymptotic expansions of the Γ-heat traces coincide.

3.4. **Lens space examples of Shams.** In [30], Shams Ul Bari studies orbifold lens spaces, orbifolds given by the quotient of the standard unit sphere by a cyclic group of isometries. Several pairs of isospectral, nonisometric orbifolds are determined. In each example, a pair of orbifolds O_1 and O_2 is given of the form $G_1 \ltimes \mathbb{S}^n$ and $G_2 \ltimes \mathbb{S}^n$, respectively, where G_1 and G_2 are cyclic groups of the same order acting as isometries on $\mathbb{S}^n$. In every example, the singular sets of O_1 and O_2 are identical, given by spheres or products of spheres with the standard metric, and the isotropy groups of these singular sets coincide. It therefore follows that the collection of twisted Γ-sectors of O_1 is isometric to the collection of twisted Γ-sectors of O_2 for any Γ, and hence each pair of isospectral lens spaces is in fact Γ-isospectral for every Γ.

4. The Sunada method and Γ-isospectrality

Early examples of isospectral pairs of manifolds were produced using ad hoc arguments. Sunada was the first to introduce a systematic method for producing isospectral manifolds, [33]. His technique is based on identifying triples (G,H_1,H_2) of finite groups, with $H_1,H_2 \leq G$, acting freely by isometries on a compact Riemannian manifold (M,g). If H_1 and H_2 are *almost conjugate* in G, meaning that each conjugacy class in G intersects H_1 and H_2 in the same number of elements, then $H_1\backslash M$ and $H_2\backslash M$ are isospectral manifolds.

In [20], Ikeda gave a simple proof of Sunada's theorem that makes it evident that the group G can be any subgroup of the group of isometries of (M,g), and that H_1 and H_2 need not act freely. (In his statement of the theorem, Ikeda assumed that H_1 and H_2 act freely, but did not use the assumption in his proof.) Thus we have the following.

Theorem 4.1 ([33], [20])**.** *Suppose that (M, g) is a compact Riemannian manifold and that G is a group that acts on (M, g) on the left by isometries. Suppose that H_1 and H_2 are finite, almost conjugate subgroups of G. Then $O_1 = H_1\backslash M$ and $O_2 = H_2\backslash M$, with their respective submersion metrics, are isospectral orbifolds.*

We remark that if H_1 and H_2 are actually conjugate in G, then the resulting orbifolds will be isometric.

In general, isospectral pairs or families of orbifolds arising from Sunada's method are not necessarily Γ-isospectral for any particular choice of Γ. We see this by noting that, as explained in Section 3, none of the pairs of isospectral orbifolds in Examples 2.7, 2.8, 2.9, 3.7, or 3.9 in [28], all of which arise from an application of Sunada's theorem, are $\mathbb{Z}^\ell$-isospectral for any positive ℓ. Similarly, although the orbifolds in any family of isospectral orbifolds constructed by Shams, Stanhope, and Webb in [31] are isospectral via Sunada's theorem, they are pairwise not $\mathbb{Z}$-isospectral as demonstrated in Section 3.1.

For given finitely generated discrete group Γ, in order to conclude that two Sunada-isospectral orbifolds are Γ-isospectral, we have the following.

Theorem 4.2. *Let (M, g) be a compact Riemannian manifold and G a group acting on (M, g) on the left by isometries. Let H_1 and H_2 be almost conjugate finite subgroups of G, and suppose that Γ is a finitely generated discrete group. If there is a bijective correspondence between homomorphisms $\varphi\colon \Gamma \to H_1$ whose images have nonempty fixed point sets and homomorphisms $\psi\colon \Gamma \to H_2$ such that for each pair φ, ψ there is an isometry $i\colon M^{\langle\varphi\rangle} \to M^{\langle\psi\rangle}$ and $C_{H_2}(\psi)$ is almost conjugate to $iC_{H_1}(\varphi)i^{-1}$ in the isometry group of $M^{\langle\psi\rangle}$, then $H_1\backslash M$ and $H_2\backslash M$ are Γ-isospectral.*

Proof. Since H_1 and H_2 are almost conjugate, by Theorem 4.1, $H_1\backslash M$ and $H_2\backslash M$ are isospectral. Since $i\colon M^{\langle\varphi\rangle} \to M^{\langle\psi\rangle}$ is an isometry, also by Theorem 4.1, $C_{H_1}(\varphi)\backslash M^{\langle\varphi\rangle}$ and $C_{H_2}(\psi)\backslash M^{\langle\psi\rangle}$ are isospectral orbifolds. Thus, there is a bijective, isospectral correspondence between the sectors of $H_1\backslash M$ and $H_2\backslash M$, so by definition of the Γ-spectrum, $H_1\backslash M$ and $H_2\backslash M$ are Γ-isospectral. $\square$

Remark 4.3. We note that since pairs of orbifolds arising from Theorem 4.1 are p-isospectral (i.e. are isospectral for the Laplace operator acting on p-forms) for all p, orbifolds arising from Theorem 4.2 are Γ-isospectral on p-forms for all p as well.

We now construct a pair of orbifolds that have nontrivial $\mathbb{Z}$- and $\mathbb{Z}^2$-sectors that are Γ-isospectral for all Γ by Theorem 4.2.

Example 4.4. Let K_1 and K_2 denote the subgroups of $SO(6)$ defined in Section 3.2, Equations 3.1 and 3.2. Recall that K_1 and K_2 are almost conjugate but not conjugate in $SO(6)$. For $i = 1, 2$, let $K_i^{\Delta_2}$ denote the subgroup of $SO(12)$ isomorphic to K_i given by identifying K_i with the diagonal in $K_i \times K_i < SO(12)$.

We define the orbifolds O_1 and O_2 as biquotients of $SO(15)$. To begin, identify $SO(3)$ with the subgroup of $SO(15)$ consisting of matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & I_{12} \end{bmatrix}$$

where $A \in SO(3)$. Similarly, identify $SO(12)$ with the subgroup of $SO(15)$ of matrices

$$\begin{bmatrix} I_3 & 0 \\ 0 & B \end{bmatrix}$$

where $B \in SO(12)$. Using this identification, we may think of $K_i^{\Delta_2} < SO(12)$ as a subgroup of $SO(15)$. Let $G_i < SO(15)$, $i = 1, 2$, be the subgroup isomorphic to $\mathbb{Z}_2^5$ generated by a_{12}, a_{23}, and $K_i^{\Delta_2}$, and note that the $K_i^{\Delta_2}$ act on coordinates 4 through 15. Furthermore, G_1 and G_2 are almost conjugate but not conjugate in $SO(15)$ for the same reason that K_1 and K_2 are almost conjugate but not conjugate in $SO(6)$.

Let $M = SO(15)/SO(3)$. Then M can be identified with the set of 12-frames in $\mathbb{R}^{15}$, where the 12-frame associated to the coset $bSO(3)$ of $b \in SO(15)$ is given by the last 12 columns of b. For $i = 1, 2$, let O_i be the orbifold presented by $G_i \ltimes M$ equipped with the submersion metric arising from a fixed biinvariant metric on $SO(15)$. By Theorem 2.5 in [28], O_1 and O_2 are isospectral orbifolds.

We now compute the sectors of the orbifolds O_1 and O_2. Every element of G_i is diagonal with eigenvalues 1 or -1. If we identify an element $bSO(3) \in M$ with a 15×12 matrix having orthonormal columns, the left action of an element $h \in G_i$ on $bSO(3)$ negates the rows in $bSO(3)$ corresponding to the positions of -1 on the diagonal in h. Thus, for an element $bSO(3) \in M$ to be fixed by h, $bSO(3)$ must have a zero row corresponding to the placement of each -1 in h. For any $h \in G_i$, we then see that $M^{\langle h \rangle}$ is the set of 12-frames in $\mathbb{R}^{15-m}$ where m is the multiplicity of -1 as an eigenvalue of h. This implies that in order for an element of G_i to have a nonempty fixed point set, it must have eigenvalue -1 with multiplicity no more than 3. We also note that by construction, only even m occur.

Since every element of $K_i^{\Delta_2}$ has eigenvalue -1 with multiplicity of at least 4, no element of $K_i^{\Delta_2}$ has nonempty fixed point set in M. Therefore, only the elements a_{12}, a_{13}, and a_{23} have nonempty fixed point sets. Hence, the only subgroups of G_i that have nonempty fixed point set in M are $\langle a_{12} \rangle$, $\langle a_{13} \rangle$, $\langle a_{23} \rangle$, and $\langle a_{12}, a_{13} \rangle \cong (\mathbb{Z}_2)^2$. Note that $M^{\langle a_{12} \rangle}$, $M^{\langle a_{13} \rangle}$, and $M^{\langle a_{23} \rangle}$ correspond to the collection of 12-frames in $\mathbb{R}^{13}$, while $M^{\langle a_{12}, a_{13} \rangle}$ corresponds to the collection of 12-frames in $\mathbb{R}^{12}$. It follows that for any finitely generated discrete group Γ, the bijection between homomorphisms $\varphi\colon \Gamma \to G_1$ and $\psi\colon \Gamma \to G_2$ with nonempty fixed point set required in Theorem 4.2 is trivial, as is the isometry $i\colon M^{\langle \varphi \rangle} \to M^{\langle \psi \rangle}$. Then as $C_{G_1}(\varphi) = G_1$ and $C_{G_2}(\psi) = G_2$ are almost conjugate, it follows that O_1 and O_2 are Γ-isospectral for all Γ. Note that as $\langle a_{12}, a_{13} \rangle$ is not a homomorphic image of $\mathbb{Z}$, both O_1 and O_2 have $\mathbb{Z}^2$-sectors that do not appear as $\mathbb{Z}$-sectors.

To show that O_1 and O_2 are not isometric, note that the lowest-dimensional $\mathbb{Z}^2$-sectors of both O_1 and O_2 are the 66-dimensional sectors corresponding to homomorphisms with image $\langle a_{12}, a_{13} \rangle$. They are given by $K_i^{\Delta_2} \backslash SO(12)$ for $i = 1, 2$, respectively. Hence, it will be sufficient to show that $K_1^{\Delta_2} \backslash SO(12)$ is not isometric to $K_2^{\Delta_2} \backslash SO(12)$.

Suppose for contradiction that $K_1^{\Delta_2} \backslash SO(12)$ and $K_2^{\Delta_2} \backslash SO(12)$ are isometric, and consider the biquotients $O_1' = K_1^{\Delta_2} \backslash SO(12)/K_1^{\Delta_2}$ and $O_2' = K_2^{\Delta_2} \backslash SO(12)/K_1^{\Delta_2}$. Then the manifold $K_1^{\Delta_2} \backslash SO(12) = K_2^{\Delta_2} \backslash SO(12)$ is by hypothesis a common Riemannian cover for both O_1' and O_2'. By [28, Corollary 2.6], O_1' and O_2' are isospectral and have different maximal isotropy orders. In fact, computations similar to those in Section 3.2.1 demonstrate that both are noneffective orbifolds with generic isotropy $\mathbb{Z}_2$. The orbifold O_1' contains points with isotropy $(\mathbb{Z}_2)^3$ that form strata of the singular set of dimension 18, while the lowest-dimensional strata of the singular set of O_2' are of dimension 34.

To see this, for each $g \in K_i$, let $\widetilde{g}$ denote the corresponding element of $K_i^{\Delta_2}$. We indicate these elements using coordinates in $SO(12)$ rather than $SO(15)$ for simplicity. Then the element $\widetilde{-I}$ acts trivially (on the right) on both $K_1^{\Delta_2}\backslash SO(12)$ and $K_2^{\Delta_2}\backslash SO(12)$. Similarly, $\widetilde{a_{12}} \in K_1^{\Delta_2}$ fixes in both $K_1^{\Delta_2}\backslash SO(12)$ and $K_2^{\Delta_2}\backslash SO(12)$ components isometric to the set of right K_i-cosets of matrices whose four 6×6 blocks are of the form

$$\left[\begin{array}{c|c} *_{2\times 2} & 0_{2\times 4} \\ \hline 0_{4\times 2} & *_{4\times 4} \end{array}\right].$$

This set of matrices in $SO(12)$ is diffeomorphic to $SO(4) \times SO(8)$, which is of dimension 34, and finitely covers the corresponding singular set in each orbifold. In O_2', by an argument similar to the one given in [28, Example 2.8], these sets have maximal isotropy $\langle \widetilde{I}, \widetilde{a_{12}} \rangle$ and hence are the lowest-dimensional singular strata. However, in O_1', the entire group $K_1^{\Delta_2}$ fixes components corresponding to matrices whose four 6×6 blocks are of the form

$$\left[\begin{array}{ccc|c} * & 0 & 0 & \\ 0 & * & 0 & 0_{3\times 3} \\ 0 & 0 & * & \\ \hline & 0_{3\times 3} & & *_{3\times 3} \end{array}\right].$$

Each such set of matrices is diffeomorphic to $SO(2)^3 \times SO(6)$, which has dimension 18, and finitely covers the lowest-dimensional singular strata in O_1'.

We conclude that O_1' and O_2' are isospectral orbifolds with a common Riemannian cover such that the lowest-dimensional singular strata in each are of different dimensions. This yields a contradiction when we demonstrate the following.

Lemma 4.5. *Suppose O_1' and O_2' are isospectral orbifolds that have as a common Riemannian cover the smooth manifold M. Then the dimensions of the lowest-dimensional singular strata of O_1' and O_2' coincide.*

Note that the orbifolds O_1' and O_2' are required to be covered by a manifold and hence are good orbifolds. The proof is similar to that of [18, Proposition 3.4(ii)].

Proof. Because O_1' and O_2' are isospectral, they have the same volume. In the expression of the asymptotic expansion of the heat kernel given in Equation 2.2, the a_k depend only on the volume of the orbifold and the curvature of M, so that the a_k coincide for O_1' and O_2'. It follows that the second terms in Equation 2.2 must coincide as well, i.e.

$$\sum_{N_1' \in \mathcal{S}(O_1')} (4\pi t)^{-\dim(N_1')/2} \sum_{k=0}^{\infty} b_{k,N_1'} t^k = \sum_{N_2' \in \mathcal{S}(O_2')} (4\pi t)^{-\dim(N_2')/2} \sum_{k=0}^{\infty} b_{k,N_2'} t^k,$$

where the sums are again over the singular strata of the orbifolds and the $b_{k,N_i'}$ are the coefficients for the strata of O_i'. However, as $b_{0,N_i'} \neq 0$ for each $N_i' \in \mathcal{S}(O_i')$, $i = 1, 2$, and since the lowest-degree terms must coincide, the claim follows. $\square$

Remark 4.6. By [16, Proposition 3.2], the Γ-sectors of a product orbifold $O \times O'$ are given by the products of the sectors of O and O'. Clearly, if O_1 and O_2 satisfy the hypotheses of Theorem 4.1, then so do $O \times O_1$ and $O \times O_2$ for any fixed (quotient) orbifold O. Therefore, by taking the product of the orbifolds in Example 4.4 with an orbifold O that has $\mathbb{Z}^\ell$-sectors that do not appear as $\mathbb{Z}^{\ell-1}$-sectors, the resulting orbifolds are Γ-isospectral for all Γ and have $\mathbb{Z}^\ell$-sectors that do not appear as $\mathbb{Z}^{\ell-1}$-sectors.

In the next example, we consider the isospectral deformation of orbifolds found in [26]. We recall that these orbifold were found using the following generalization of the Sunada theorem in [6], recast in the orbifold setting in [26]. We will show that any pair of orbifolds in the deformation are Γ-isospectral for any Γ. While we will not be able to prove this using a direct application of Theorem 4.2 because the groups involved are not finite, the philosophy will be the same. We will show that there is a bijection between Γ-sectors such that corresponding sectors are isospectral; we will in fact show that they are isometric.

For a Lie group G with subgroup H, we say that an automorphism $\Phi\colon G \to G$ is an *almost-inner automorphism relative to* H if for all $h \in H$ there is an element $a \in G$ such that $\Phi(h) = aha^{-1}$.

Theorem 4.7 ([6], [26]). *Suppose that G is a Lie group with simply connected, nilpotent identity component G_0. Let H be a discrete subgroup of G such that $G = HG_0$ and $(G_0 \cap H)\backslash G_0$ is compact. Suppose that G acts effectively and properly discontinuously on the left by isometries on (M, g) with $H\backslash M$ compact. Let $\Phi\colon G \to G$ be an almost-inner automorphism relative to H. Then, letting g denote the submersion metric, the quotient orbifolds $(H\backslash M, g)$ and $(\Phi(H)\backslash M, g)$ are isospectral.*

Example 4.8. Let G be the Lie group

$$\{(x_1, x_2, y_1, y_2, z_1, z_2) \,|\, x_i, y_i, z_i \in \mathbb{R}\}$$

with group multiplication given by

$$(x_1, \dots, z_2)(x_1', \dots, z_2') \\ = (x_1 + x_1', \dots, y_2 + y_2', z_1 + z_1' + x_1y_1' + x_2y_2', z_2 + z_2' + x_1y_2').$$

We denote elements of $\mathrm{Aut}(G) \ltimes G$ by ordered pairs $(\psi, \bar{x})$. The group multiplication in $\mathrm{Aut}(G) \ltimes G$ is given by $(\psi, \bar{x})(\psi', \bar{x}') = (\psi\psi', \bar{x}\psi(\bar{x}'))$ and $\mathrm{Aut}(G) \ltimes G$ acts on G by $(\psi, \bar{x}) \cdot \bar{g} = \bar{x}\psi(\bar{g})$.

Suppose that Λ is the integer lattice in G. Let $\alpha \in \mathrm{Aut}(G) \ltimes G$ be given by the ordered pair $(\varphi, (0, 0, 0, 0, 0, \frac{1}{2}))$, where φ is the element of $\mathrm{Aut}(G)$ given by $\varphi(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1, x_2, -y_1, -y_2, -z_1, -z_2)$. Then

$$\alpha(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1, x_2, -y_1, -y_2, -z_1, -z_2 + \tfrac{1}{2}).$$

Define $H = \Lambda \cup \alpha\Lambda$.

For $t \in [0, 1)$ define an automorphism $\Phi_t\colon G \to G$ by

$$\Phi_t(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1, x_2, y_1, y_2, z_1, z_2 + ty_2).$$

From [6], Φ_t is an almost-inner automorphism of G relative to Λ. In particular, letting $a = (0, 0, 0, 0, 0, 0)$ if $y_2 = 0$ and $(t, -\frac{ty_1}{y_2}, 0, 0, 0, 0)$ otherwise, we see that $\Phi_t(x) = axa^{-1}$ for any $x \in G$. Recalling that $HG = G \cup \alpha G$, extend Φ_t to an automorphism $\tilde{\Phi}_t\colon HG \to HG$ by setting $\tilde{\Phi}_t(x) = \tilde{\Phi}_t(\mathrm{Id}, x) = (\mathrm{Id}, \Phi_t(x))$ and $\tilde{\Phi}_t(\alpha x) = \tilde{\Phi}_t(\varphi, \alpha \cdot x) = (\varphi, \Phi_t(\alpha \cdot x)) = (\varphi, \alpha \cdot \Phi_t(x))$. Then $\tilde{\Phi}_t$ is an almost-inner automorphism relative to H; indeed, for $h = x$ or $\alpha x \in H$, $\tilde{\Phi}_t(h) = (\mathrm{Id}, a)h(\mathrm{Id}, a^{-1})$ where a is based on x as above. Therefore, letting g be an HG-invariant metric on G, by Theorem 4.7, we have a continuous isospectral family of orbifolds, $(\tilde{\Phi}_t(H)\backslash G, g)$, $t \in [0,1)$. By [26], the deformation is nontrivial.

We now show that this deformation is Γ-isospectral for all Γ. We begin by computing the sectors of $(\tilde{\Phi}_t(H)\backslash G, g)$. Since $H = \Lambda \cup \alpha\Lambda$, $\tilde{\Phi}_t(H) = \tilde{\Phi}_t(\Lambda) \cup \tilde{\Phi}_t(\alpha\Lambda)$. Elements of $\tilde{\Phi}_t(\Lambda)$ are elements of G and thus have empty fixed point sets. Consider $\tilde{\Phi}_t(\alpha\lambda)$ where $\lambda \in \Lambda$. If $\lambda = (a,b,c,d,e,f)$, then as an ordered pair in $\mathrm{Aut}(G) \ltimes G$,

$$\tilde{\Phi}_t(\alpha\lambda) = (\varphi, (a, b, -c, -d, -e, -f + \tfrac{1}{2} - td)).$$

For $x = (x_1, x_2, y_1, y_2, z_1, z_2) \in G$, direct computation shows that $\tilde{\Phi}_t(\alpha\lambda) \cdot x = x$ if and only if $a = b = 0$, $c = -2y_1$, $d = -2y_2$, $e = -2z_1$, and $f = -2z_2 + \frac{1}{2} - td$. In this case, for $\lambda = (0,0,c,d,e,f)$ with $c,d,e,f \in \mathbb{Z}$, the fixed point set of $\tilde{\Phi}_t(\alpha\lambda)$ is

$$G^{\langle \tilde{\Phi}_t(\alpha\lambda)\rangle} = \{(x_1, x_2, -\tfrac{c}{2}, -\tfrac{d}{2}, -\tfrac{e}{2}, -\tfrac{f}{2} + \tfrac{1}{4} - \tfrac{td}{2}) \,|\, x_1, x_2 \in \mathbb{R}\}.$$

We note that if $\lambda = (0,0,c,d,e,f)$, then the order of $\tilde{\Phi}_t(\alpha\lambda)$ in $\mathrm{Aut}(G) \ltimes G$ is 2. Moreover, for $\lambda \neq \lambda'$, the fixed point sets of $\tilde{\Phi}_t(\alpha\lambda)$ and $\tilde{\Phi}_t(\alpha\lambda')$ do not intersect. Therefore, the only nontrivial isotropy groups in $\tilde{\Phi}_t(H)\backslash G$ are $\{\mathrm{Id}, \tilde{\Phi}_t(\alpha\lambda)\} \cong \mathbb{Z}_2$. This implies that if Γ is a group that admits $\mathbb{Z}_2$ as a homomorphic image, the Γ-sectors of $\tilde{\Phi}_t(H)\backslash G$ will all be of the form $C_{\tilde{\Phi}_t(H)}(\tilde{\Phi}_t(\alpha\lambda))\backslash G^{\langle \tilde{\Phi}_t(\alpha\lambda)\rangle}$. If Γ does not admit $\mathbb{Z}_2$ as a homomorphic image, $\tilde{\Phi}_t(H)\backslash G$ has no nontrivial Γ-sectors.

For $t \in [0,1)$ and for fixed $\lambda = (0,0,c,d,e,f)$, the action of the element $i = (\mathrm{Id}, (0,0,0,0,0,-\frac{td}{2})) \in \mathrm{Aut}(G) \ltimes G$ maps $G^{\langle \alpha\lambda\rangle}$ to $G^{\langle \tilde{\Phi}_t(\alpha\lambda)\rangle}$. Since $G^{\langle \tilde{\Phi}_t(\alpha\lambda)\rangle}$ is a totally geodesic submanifold of G, the metric on $G^{\langle \tilde{\Phi}_t(\alpha\lambda)\rangle}$ is given by the restriction of the metric from G. Since $i \in G < HG$, and the metric on G is HG-invariant, i is an isometry from $G^{\langle \alpha\lambda\rangle}$ to $G^{\langle \tilde{\Phi}_t(\alpha\lambda)\rangle}$.

For $\lambda = (0,0,c,d,e,f)$ and $t \in [0,1)$, the centralizer of $\tilde{\Phi}_t(\alpha\lambda)$ in $\tilde{\Phi}_t(H)$ is $\{(\mathrm{Id}, (p,q,0,0,\frac{cp}{2} + \frac{dq}{2}, \frac{dp}{2}))\} \cup \{(\varphi, (p,q,-c,-d,-e-\frac{cp}{2} - \frac{dq}{2}, \frac{1}{2} - f - \frac{dp}{2} - dt))\}$ where $p,q \in \mathbb{Z}$ are such that $cp + dq$ and dp are both even. (Note that $C_H(\alpha\lambda)$ corresponds to $t = 0$.) For $i = (\mathrm{Id}, (0,0,0,0,0,\frac{-td}{2})$ as above, direct computation shows that $iC_H(\alpha\lambda)i^{-1} = C_{\tilde{\Phi}_t(H)}(\tilde{\Phi}_t(\alpha\lambda))$. Since these groups are in fact equal, for any $t \in [0,1)$, the sectors $C_H(\alpha\lambda)\backslash G^{\langle \alpha\lambda\rangle}$ and $C_{\tilde{\Phi}_t(H)}(\tilde{\Phi}_t(\alpha\lambda)\backslash G^{\langle \tilde{\Phi}_t(\alpha\lambda)\rangle}$ are isometric, hence isospectral. Therefore, the collection $\tilde{\Phi}_t(H)\backslash G$, $t \in [0,1)$, is a continuous deformation of orbifolds that are Γ-isospectral for all Γ.

For our final example, we construct a pair of 5-dimensional flat orbifolds that are Γ-isospectral for all Γ. Here again, since the groups involved are not finite, we will not be able to apply Theorem 4.2 directly but we will use the same idea that underlies the proof of that theorem: after proving that the orbifolds themselves are isospectral using the eigenvalue counting method of Miatello and Rossetti, we will show that there is a bijection between Γ-sectors such that corresponding sectors are isospectral.

Example 4.9. Let L_1 and L_2 be a pair of 4-dimensional isospectral nonisometric lattices found in Section 2 of [5]. Orthogonally extend L_1 and L_2 by vectors of the equal length to 5-dimensional isospectral, nonisometric lattices Λ_1 and Λ_2 (see [2, p.154]).

Let g be the isometry of $\mathbb{R}^5$ given by reflection across the copy of $\mathbb{R}^4$ that contains L_1 and L_2. For $i = 1,2$, let G_i be the subgroup of $O(5) \ltimes \mathbb{R}^5$ generated by g and Λ_i. Letting $T_{\Lambda_i} = \Lambda_i\backslash\mathbb{R}^5$ and $O_i = G_i\backslash\mathbb{R}^5$, T_{Λ_i} covers O_i and $O_i \cong \overline{G_i}\backslash T_{\Lambda_1}$ where

$\overline{G_i} := G_i/\Lambda_i$. (See [28, p.357].) Letting F denote the projection of G_i onto $O(5)$, by the first isomorphism theorem, $F = \{\mathrm{Id}, g\}$ is isomorphic to $\overline{G}_i$. Thus we can identify O_i with $\mathbb{Z}_2\backslash T_{\Lambda_i}$.

We confirm that O_1 and O_2 are isospectral using Miatello and Rosetti's eigenvalue counting formula, Theorem 3.1 in [28]. For any eigenvalue μ, the multiplicity of μ in the spectrum of O_i is given by

$$d_\mu(G_i) = (\#F)^{-1} \sum_{B\in F} e_{\mu,B}(G_i), \text{ where } e_{\mu,B}(G_i) := \sum_{v\in\Lambda_i^*, \|v\|^2=\mu, Bv=v} e^{2\pi i\langle v,b\rangle}$$

and b is chosen so that $BL_b \in G_i$.

It is straightforward to compute $d_\mu(G_i)$. When $B = \mathrm{Id}$, let b be any element of Λ_i. Since $\langle v, b\rangle \in \mathbb{Z}$ for all $v \in \Lambda_i^*$,

$$e_{\mu,\mathrm{Id}}(G_i) = \#\{v \in \Lambda_i^* \mid \|v\|^2 = \mu\}$$

i.e. $e_{\mu,\mathrm{Id}}(G_i)$ is the multiplicity of μ as an eigenvalue of T_{Λ_i}. Since T_{Λ_1} and T_{Λ_2} are isospectral, $e_{\mu,\mathrm{Id}}(G_1) = e_{\mu,\mathrm{Id}}(G_2)$.

When $B = g$, note that the elements of Λ_i^* that are fixed by B are the elements of L_i. Thus $e_{\mu,g}(G_i)$ is equal to the multiplicity of μ as an eigenvalue of $L_i\backslash\mathbb{R}^4$. Since $L_1\backslash\mathbb{R}^4$ and $L_2\backslash\mathbb{R}^4$ are isospectral, $e_{\mu,g}(G_1) = e_{\mu,g}(G_2)$ for all μ.

Therefore for any eigenvalue μ, $d_\mu(G_1) = d_\mu(G_2)$ so $G_1\backslash\mathbb{R}^5$ and $G_2\backslash\mathbb{R}^5$ are isospectral orbifolds.

We now confirm that these orbifolds are Γ-isospectral for any Γ. Since for $i = 1, 2$ the lattice Λ_i acts on $\mathbb{R}^5$ by translation, the only finite subgroups of G_i are of the form $\{\mathrm{Id}, (g, (0, 0, 0, 0, e))\} \cong \mathbb{Z}_2$ where $(0, 0, 0, 0, e) \in \Lambda_i$. Thus for any finitely generated discrete group Γ, either Γ admits $\mathbb{Z}_2$ as a homomorphic image or it does not. If it does, O_i will have a nontrivial Γ-sector for each nontrivial $\varphi \in \mathrm{HOM}(\Gamma, G_i)$ having image $\{\mathrm{Id}, (g, (\bar{0}, e))\}$. Since Λ_1 and Λ_2 are extensions of L_1 and L_2 by the same vector in $\mathbb{R}^5$ which is orthogonal to both L_1 and L_2, for any homomorphism $\varphi\colon \Gamma \to G_1$, there is an obvious corresponding homomorphism $\psi\colon \Gamma \to G_2$ that has the same image as φ.

If $\varphi(\Gamma) = \{\mathrm{Id}, (g, (\bar{0}, e))\}$, then the fixed point set of the image $\varphi(\Gamma)$ of φ is $(\mathbb{R}^5)^{\langle\varphi\rangle} = \{(x, y, z, w, \frac{e}{2}) \mid x, y, z, w \in \mathbb{R}\}$. The centralizer of $\varphi(\Gamma)$ in G_i is $C_{G_i}(\varphi) = \{(\mathrm{Id}, (\bar{a}, 0)) \mid \bar{a} \in L_i\} \cup \{(g, (\bar{a}, e)) \mid \bar{a} \in L_i\}$. A typical Γ-sector in O_i is of the form $C_{G_i}(\varphi)\backslash(\mathbb{R}^5)^{\langle\varphi\rangle}$.

We note that for $i = 1$ or 2, all of the Γ-sectors of O_i are isometric to each other. Indeed if $\varphi(\Gamma) = \{\mathrm{Id}, (g, (0, 0, 0, 0, 0))\}$, then for any other homomorphism $\varphi'\colon \Gamma \to G_i$ with image $\{\mathrm{Id}, (g, (\bar{0}, e))\}$, translation by $p = (\bar{0}, \frac{e}{2})$ is an isometry from $(\mathbb{R}^5)^{\langle\varphi\rangle}$ to $(\mathbb{R}^5)^{\langle\varphi'\rangle}$ and $L_p C_{G_i}(\varphi) L_p^{-1} = C_{G_i}(\varphi')$. Therefore the sectors $C_{G_i}(\varphi)\backslash(\mathbb{R}^5)^{\langle\varphi\rangle}$ and $C_{G_i}(\varphi')\backslash(\mathbb{R}^5)^{\langle\varphi'\rangle}$ are isometric.

Finally, for any choice of Γ that admits $\mathbb{Z}_2$ as a homomorphic image, the corresponding Γ-sectors of O_1 are O_2 are isospectral as follows. Suppose that $\varphi\colon \Gamma \to G_1$ has image $\{\mathrm{Id}, (g, (0, 0, 0, 0, 0))\}$. The corresponding homomorphism $\psi\colon \Gamma \to G_2$ has the same image. The images of both φ and ψ also have the same fixed point sets, namely the copy of $\mathbb{R}^4$ fixed by the action of g on $\mathbb{R}^5$. Recalling that $F = \{\mathrm{Id}, g\}$, the centralizer of $\varphi(\Gamma)$ in G_1 is given by $F \ltimes L_1$ and the centralizer of $\psi(\Gamma)$ in G_2 is given by $F \ltimes L_2$. Since L_1 and L_2 are isospectral, by an argument similar to the one given above using Miatello and Rossetti's eigenvalue counting formula, $(F \ltimes L_1)\backslash\mathbb{R}^4$ is isospectral to $(F \ltimes L_2)\backslash\mathbb{R}^4$. Since all other sectors in O_i for $i = 1$

or 2 are isometric to these sectors respectively, we conclude that O_1 and O_2 are Γ-isospectral.

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