PH 500 Problem Set #6

In the last problem set, we put together two spin-1/2 particles and obtained one state with spin 0 and three with spin 1, where by "spin j" we meant that the eigenvalue of the total angular momentum squared was $\hbar^2 j(j+1)$. Now we'd like to work with more general angular momenta. Again, what you need to do is in **bold**.

All forms of angular momentum obey the same commutation relations as spin,

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \qquad [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x \qquad [\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y \tag{1}$$

where $\hat{\boldsymbol{J}}$ can stand for any form of angular momentum (a single spin, the total of several spins, etc.). We define

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \tag{2}$$

and

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \tag{3}$$

From the commutation relations in Eq. (1) we can derive

$$[\hat{J}_k, \hat{J}^2] = 0 \qquad \text{for all } k = x, y, z, +, - \tag{4}$$

and

$$[\hat{J}_z, \hat{J}_\pm] = \pm\hbar \hat{J}_\pm \qquad \text{and} \qquad [\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z \tag{5}$$

and also two expressions for $\hat{J}^2$:

$$\hat{J}^2 = \hat{J}_\pm \hat{J}_\mp + \hat{J}_z^2 \mp \hbar \hat{J}_z \tag{6}$$

Now let's put all of this together. Since $\hat{J}_z$ and $\hat{J}^2$ commute, we can form a simultaneous basis of eigenstates of both operators. Suppose we have such a simultaneous eigenstate, with eigenvalue ρ for $\hat{J}^2$ and λ for $\hat{J}_z$:

$$\begin{aligned} \hat{J}^2|\psi\rangle &= \rho|\psi\rangle \\ \hat{J}_z|\psi\rangle &= \lambda|\psi\rangle \end{aligned} \tag{7}$$

Now consider $\hat{J}_+|\psi\rangle$. Using the commutation relations above,

$$\begin{aligned} \hat{J}^2(\hat{J}_+|\psi\rangle) = \hat{J}_+\hat{J}^2|\psi\rangle = \rho(\hat{J}_+|\psi\rangle) \\ \hat{J}_z(\hat{J}_+|\psi\rangle) = \hat{J}_+(\hat{J}_z + \hbar)|\psi\rangle = (\lambda + \hbar)(\hat{J}_+|\psi\rangle) \end{aligned} \tag{8}$$

Thus we see that $\hat{J}_+|\psi\rangle$ is an eigenvector of $\hat{J}^2$ with the same eigenvalue ρ, and it is an eigenvector of $\hat{J}_z$ with eigenvalue $\lambda + \hbar$. As we saw in the special case before, the eigenstate has been "raised" by one unit of $\hbar$. Similarly, $\hat{J}_-|\psi\rangle$ is an eigenvector of $\hat{J}^2$ with the same eigenvalue ρ, and it is an eigenvector of $\hat{J}_z$ with eigenvalue $\lambda - \hbar$.

Thus we can raise and lower the $\hat{J}_z$ eigenvalue by repeated application of $\hat{J}_+$ or $\hat{J}_-$. However, this process has to terminate. We can't keep raising the $\hat{J}_z$ eigenvalue forever, because $\hat{J}_z^2$ should never exceed $\hat{J}^2$ — after all, $\hat{J}^2$ is equal to $\hat{J}_z^2$ plus nonnegative terms. The only way the process can end is if the raising or lowering operator yields a *zero* vector — which satisfies Eq. (8) trivially.

At the top and bottom, we have

$$\begin{aligned} \hat{J}_+|\rho \quad \lambda_t\rangle &= 0 \\ \hat{J}_-|\rho \quad \lambda_b\rangle &= 0 \end{aligned} \tag{9}$$

Now let's apply $\hat{J}^2$ to these states, using Eq. (6). We have

$$\begin{aligned} \hat{J}^2|\rho \quad \lambda_t\rangle &= (\hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar\hat{J}_z)|\rho \quad \lambda_t\rangle = \lambda_t(\lambda_t+\hbar)|\rho \quad \lambda_t\rangle = \rho|\rho \quad \lambda_t\rangle \\ \hat{J}^2|\rho \quad \lambda_b\rangle &= (\hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z)|\rho \quad \lambda_b\rangle = \lambda_b(\lambda_b-\hbar)|\rho \quad \lambda_b\rangle = \rho|\rho \quad \lambda_b\rangle \end{aligned} \tag{10}$$

Thus we have

$$\rho = \lambda_t(\lambda_t+\hbar) = \lambda_b(\lambda_b-\hbar) \tag{11}$$

and we know that λ_t and λ_b differ by an integer number of units of $\hbar$, so we let $\lambda_t = \lambda_b + \hbar n$ where $n \in 0, 1, 2, 3 \ldots$. The we have

$$\begin{aligned} (\lambda_b+\hbar n)(\lambda_b+\hbar(n+1)) &= \lambda_b(\lambda_b-\hbar) \\ \lambda_b^2 + \lambda_b\hbar(2n+1) + \hbar^2 n(n+1) &= \lambda_b^2 - \lambda_b\hbar \\ 2\lambda_b\hbar(n+1) &= -\hbar^2 n(n+1) \\ \lambda_b &= -\hbar\frac{n}{2} \end{aligned} \tag{12}$$

Let $j = n/2$, so that $\lambda_t = -\lambda_b = \hbar j$ and $\rho = \hbar^2 j(j+1)$. Putting all of these resgults together, we find that we could choose eigenstates of the total angular momentum $\hat{J}^2$ and the z-component $\hat{J}_z$. These states are $|j \quad m\rangle$ with

$$\hat{J}^2|j \quad m\rangle = \hbar^2 j(j+1)|j \quad m\rangle \qquad \text{and} \qquad \hat{J}_z|j \quad m\rangle = \hbar m|j \quad m\rangle \tag{13}$$

where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2\ldots$ and $m = -j, -j+1, \ldots, j-1, j$. For a given j, there are thus a total of $2j+1$ states.

There is one more detail to fill in. We know that $\hat{J}_+|j \quad m\rangle$ is an eigenvector of $\hat{J}^2$ and $\hat{J}_z$ with eigenvalues $\hbar^2 j(j+1)$ and $\hbar(m+1)$ respectively. However, we *cannot* conclude that this result is *equal* to $|j \quad m+1\rangle$ because it may be normalized differently. In fact, if we write

$$\begin{aligned} \hat{J}_+|j \quad m\rangle &= C^+_{jm}|j \quad m+1\rangle \\ \hat{J}_-|j \quad m\rangle &= C^-_{jm}|j \quad m-1\rangle \end{aligned} \tag{14}$$

then for $|j \quad m+1\rangle$ to be normalized we require

$$\langle j \quad m|\hat{J}_-\hat{J}_+|j \quad m\rangle = \langle j \quad m+1|(C^+_{jm})^* C^+_{jm}|j \quad m+1\rangle = |C^+_{jm}|^2 \tag{15}$$

where we have used $(\hat{J}_+)^\dagger = \hat{J}_-$. Now using Eq. (6) we have

$$\langle j \quad m|\hat{J}_-\hat{J}_+|j \quad m\rangle = \langle j \quad m|\left(\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z\right)|j \quad m\rangle = \hbar^2\left(j(j+1) - m(m+1)\right) \tag{16}$$

Furthermore, we can choose phase conventions for our states such that C^+_{jm} is always real, so we have

$$\hat{J}_+|j \quad m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}|j \quad m+1\rangle \tag{17}$$

and similarly

$$\hat{J}_-|j \quad m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}|j \quad m-1\rangle \tag{18}$$

Note that these coefficients become zero when one tries to raise past the top of the latter ($j = m$) or lower past the bottom ($j = -m$). They also agree with the results we found for the special case of $j = 1/2$.

1. Consider the case $j = 1$. **Construct the explicit matrices for $\hat{J}^2$, $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ in the basis of eigenstates of $\hat{J}^2$ and $\hat{J}_z$. Verify that your matrices obey the correct commutation relations.**

   Hint: First, what is the dimension of the space? What are the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$ in each state? Then find the matrices for $\hat{J}_+$ and $\hat{J}_-$ and go from there.

2. Consider a rigid rotator constrained to move on an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

   for which $a = b$ (this is a generalization of the case of a sphere, for which $a = b = c$). The only thing you need to know is that the Hamiltonian is

$$\hat{\mathcal{H}} = \frac{1}{2I_1}\hat{J}_z^2 + \frac{1}{2I_2}\left(\hat{J}_x^2 + \hat{J}_y^2\right)$$

   Find the eigenvalues of $\hat{\mathcal{H}}$ and their degeneracies (that is, how many states there are with that eigenvalue). Assume that I_1/I_2 is an irrational number, so that no combination of multiplication and division by integers can turn I_1 into I_2 or vice versa.

3. For the ellipsoidal rotator of the previous problem, suppose we start in the state

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|j=1 \quad m=0\rangle + |j=1 \quad m=1\rangle\right)$$

   Find the expectation values $\langle\hat{J}_x\rangle$, $\langle\hat{J}_y\rangle$, $\langle\hat{J}_z\rangle$, and $\langle\hat{J}^2\rangle$ as functions of time.

   Hint: Express $\langle\hat{J}_x\rangle$ and $\langle\hat{J}_y\rangle$ in terms of $\langle\hat{J}_+\rangle$ and $\langle\hat{J}_-\rangle$.

4. Suppose that a particle is in the state $|j \quad m\rangle$ with $\hat{J}^2|j \quad m\rangle = \hbar^2 j(j+1)|j \quad m\rangle$ and $\hat{J}_z|j \quad m\rangle = \hbar m|j \quad m\rangle$.

   (a) **Find the expectation values $\langle\hat{J}_x\rangle$, $\langle\hat{J}_y\rangle$, and $\langle\hat{J}_z\rangle$ in this state.**

   (b) **Find the expectation values $\langle\hat{J}_x^2\rangle$, $\langle\hat{J}_y^2\rangle$, and $\langle\hat{J}_z^2\rangle$ in this state.**

   (c) **Find the uncertainties ΔJ_x, ΔJ_y, and ΔJ_z in this state. For each of the three different pairings of $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$, check the corresponding uncertainty relation in this state and indicate when the minimum uncertainty is achieved.**

Next, we consider how to *add* two arbitrary angular momenta. Suppose we have one particle with angular momentum j_1 and another with angular momentum j_2 and we'd like to consider the total angular momentum. What does that mean? Well, we would first consider states like $|m_1 \;\; m_2\rangle$, in analogy to the spin case. These are eigenstates of $\hat{J}_{1z}$ (with eigenvalue $\hbar m_1$), of $\hat{J}_{2z}$ (with eigenvalue $\hbar m_2$), of $\hat{J}_1^2$ (with eigenvalue $\hbar^2 j_1(j_1+1)$), and of $\hat{J}_2^2$ (with eigenvalue $\hbar^2 j_2(j_2+1)$). This construction is possible because all four of these operators commute. To consider the *total* spin, we would like to find eigenstates of $\hat{J}^2 = (\hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_2)^2 = \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}}_2$ (we don't have to worry about ordering here since all the $\hat{\boldsymbol{J}}_1$ operators commute with all the $\hat{\boldsymbol{J}}_2$ operators). The total angular momentum operator $\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}_{\boldsymbol{1}} + \hat{\boldsymbol{J}}_{\boldsymbol{2}}$ is just another angular momentum operator, obeying the same commutation relations as spin or any other form of angular momentum, so we can enumerate the simultaneous eigenstates of $\hat{J}^2$ and $\hat{J}_z$ by the exact same procedure as above.

While $\hat{J}^2$ still commutes with $\hat{J}_1^2$ and $\hat{J}_2^2$, it does *not* commute with $\hat{J}_{1z}$ or $\hat{J}_{2z}$. It does, however, still commute with the *sum* $\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$. So our goal is to construct states $|j \;\; m\rangle$ which are eigenstates of $\hat{J}^2$ (with eigenvalue $\hbar^2 j(j+1)$), of $\hat{J}_z$ (with eigenvalue $\hbar m$), and still of $\hat{J}_1^2$ (with eigenvalue $\hbar^2 j_1(j_1+1)$), and of $\hat{J}_2^2$ (with eigenvalue $\hbar^2 j_2(j_2+1)$). We'll express these states in term of the old basis vectors $|m_1 \;\; m_2\rangle$ that are eigenstates of the two z-components individually.

We start with the easiest case. Suppose that we consider both angular momenta separately, and arrange that they are aligned as much as possible with the z-axis, so we are in the state $|m_1 = j_1 \;\; m_2 = j_2\rangle$. This state is also an eigenstate of $\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$ with eigenvalue $j_1 + j_2$. Furthermore, we have

$$\begin{aligned} \hat{J}^2|m_1 = j_1 \;\; m_2 = j_2\rangle &= \left(\hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar\hat{J}_z\right)|m_1 = j_1 \;\; m_2 = j_2\rangle \\ &= \hbar^2(j_1+j_2)(j_1+j_2+1)|m_1 = j_1 \;\; m_2 = j_2\rangle \end{aligned} \tag{19}$$

where we have used that the two terms generated by $\hat{J}_+ = \hat{J}_{1+} + \hat{J}_{2+}$ both give zero because they try to raise the eigenvalue past the top of the ladder. Therefore this state is an eigenvector of $\hat{J}^2$, with eigenvalue $\hbar^2(j_1+j_2)(j_1+j_2+1)$.

So we can write the first entry in our translation between the two bases as

$$|j = j_1 + j_2 \;\; m = j_1 + j_2\rangle = |m_1 = j_1 \;\; m_2 = j_2\rangle \tag{20}$$

Of course, this was the easy one. But with a foot in the door, we can find another state by applying the lowering operator to both sides,

$$\begin{aligned} \hat{J}_-|j = j_1 + j_2 \;\; m = j_1 + j_2\rangle &= (\hat{J}_{1-} + \hat{J}_{2-})|m_1 = j_1 \;\; m_2 = j_2\rangle \\ \hbar\sqrt{j(j+1) - m(m-1)}\times & \\ |j = j_1 + j_2 \;\; m = j_1 + j_2 - 1\rangle &= \hbar\sqrt{j_1(j_1+1) - j_1(j_1-1)}|m_1 = j_1 - 1 \;\; m_2 = j_2\rangle \\ &+ \hbar\sqrt{j_2(j_2+1) - j_2(j_2-1)}|m_1 = j_1 \;\; m_2 = j_2 - 1\rangle \\ \hbar\sqrt{2(j_1+j_2)}|j = j_1 + j_2 \;\; m = j_1 + j_2 - 1\rangle &= \hbar\sqrt{2j_1}|m_1 = j_1 - 1 \;\; m_2 = j_2\rangle \\ &+ \hbar\sqrt{2j_2}|m_1 = j_1 \;\; m_2 = j_2 - 1\rangle \\ \hbar|j = j_1 + j_2 \;\; m = j_1 + j_2 - 1\rangle &= \hbar\sqrt{\frac{j_1}{(j_1+j_2)}}|m_1 = j_1 - 1 \;\; m_2 = j_2\rangle \end{aligned}$$

$$+ \quad \hbar\sqrt{\frac{j_2}{(j_1+j_2)}}|m_1 = j_1 \;\; m_2 = j_2 - 1\rangle \tag{21}$$

giving us a (correctly normalized) expression for $|j = j_1 + j_2 \;\; m = j_1 + j_2 - 1\rangle$. The inner products

$$C^{j_1 j_2 j}_{m_1 m_2 m} = \langle m_1 \;\; m_2 | j \;\; m\rangle \tag{22}$$

describing the change of basis from the total angular momentum basis to the basis of individual angular momenta are called *Clebsch-Gordan coefficients*.

We can continue applying the lowering operator to obtain all the states with $j = j_1 + j_2$ for $m = -(j_1 + j_2) \ldots (j_1 + j_2)$, a total of $2(j_1 + j_2) + 1$ states. The last state in this sequence will be

$$|j = j_1 + j_2 \;\; m = -(j_1 + j_2)\rangle = |m = -j_1 \;\; m = -j_2\rangle \tag{23}$$

However, our new basis should comprise a total of $(2j_1+1)(2j_2+1)$ independent states, the *product* of the dimensionalities of the spaces for each angular momentum individually. Where are the rest? Well, just as in the case of two spin-1/2, what we've found so far is the case where the *magnitudes* add up — that is, the two angular momenta are aligned with each other. But the magnitude of the sum could be less than the sum of the magnitudes of the individuals if the two are not aligned.

So, can we find states with $j < j_1 + j_2$? A good place to start would be to look for $m = j = j_1 + j_2 - 1$ — the top of the next ladder. This state is an eigenstate of $\hat{J}_z$ with eigenvalue $\hbar(j_1 + j_2 - 1)$. Therefore it must be built out of a combination of the states $|m_1 = j_1 - 1 \;\; m_2 = j_2\rangle$ and $|m_1 = j_1 \;\; m_2 = j_2 - 1\rangle$ since these are the only states whose total z-component is $j_1 + j_2 - 1$. Furthermore, the state must be normalized and orthogonal to the $|j = j_1 + j_2 \;\; m = j_1 + j_2 - 1\rangle$ state we found above. There is a unique such state (up to the usual overall phase we can include in any eigenstate),

$$\begin{aligned} \hbar|j = j_1 + j_2 - 1 \;\; m = j_1 + j_2 - 1\rangle &= \hbar\sqrt{\frac{j_2}{(j_1+j_2)}}|m_1 = j_1 - 1 \;\; m_2 = j_2\rangle \\ &- \hbar\sqrt{\frac{j_1}{(j_1+j_2)}}|m_1 = j_1 \;\; m_2 = j_2 - 1\rangle \end{aligned} \tag{24}$$

Now iterate this procedure: We can lower this state repeatedly to obtain the $2(j_1 + j_2 - 1) + 1$ states of different m for this j. We then find the unique normalized state orthogonal to both $|j = j_1 + j_2 \;\; m = j_1 + j_2 - 2\rangle$ and $|j = j_1 + j_2 - 1 \;\; m = j_1 + j_2 - 2\rangle$, which must be equal to $|j = j_1 + j_2 - 2 \;\; m = j_1 + j_2 - 2\rangle$. Then this state starts the ladder for $j = j_1 + j_2 - 2$, and so on.

When does the process end? When $j = |j_2 - j_1|$, there are no more orthogonal states available to start a new ladder, and we are finished. How many states did we get altogether? The total is

$$\sum_{j=|j_2-j_1|}^{j=j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1) \tag{25}$$

so we found them all!

We note that for the special case of $j_1 = j_2 = 1/2$, the states we have displayed explicitly, eqs. (20), (21), (23) and (24), correspond exactly to the four states we found before.

5. Suppose we have two particles with angular momenta $j_1 = 1$ for the first particle and $j_2 = 2$ for the second particle. We measure the total angular momentum operators $\hat{J}^2$ and $\hat{J}_z$ and obtain $12\hbar^2$ and $2\hbar$ respectively.

   (a) First, we immediately measure the z-component of the first particle's angular momentum. **Give the possible outcomes of this measurement, together with the probability of each.**

   (b) Immediately after performing the measurement in part (a), we measure the z-component of the second particle's angular momentum. **For each possible result in (a), give the possible outcomes of this subsequent measurement, together with the probability of each outcome.**

   (c) Pick one of the possible outcomes in (b) and suppose that we then immediately measure both $\hat{J}^2$ and $\hat{J}_z$. **Find the possible outcomes of these measurements and the probabilities of each.**

   Hint: you need to add the angular momenta here (so that you can pass between the coupled and uncoupled bases), but you only need a few of the states in the coupled basis.

6. Consider an electron (which has internal spin $s = 1/2$) in an $\ell = 1$ orbital angular momentum state. It also has a fixed radial wavefunction which we can ignore for the purposes of this problem.

   Suppose the Hamiltonian is given by

$$\hat{\mathcal{H}} = -\alpha \hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{L}} - \beta \left(\hat{S}_z + \hat{L}_z \right) \tag{26}$$

   (we're ignoring the energy associated with its radial motion since that is held fixed throughout the problem).

   The first term represents a spin-orbit coupling: the electron's internal magnetic dipole moment and the dipole moment created by its orbital motion like to align with each other. The second term would be the effect of an applied magnetic field in the $\hat{\boldsymbol{z}}$ direction: both dipole moments like to align with the field.

   (a) **Find the allowed eigenvalues of $\hat{\mathcal{H}}$. Construct the corresponding eigenstates in terms of the basis vectors $|s \;\; \ell \;\; m_s \;\; m_\ell\rangle$ where $s = 1/2$, $\ell = 1$, $m_s = \pm 1/2$ and $m_\ell = -1, 0, 1$.**

   Hint: Write the Hamiltonian in terms of total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$.

   (b) At $t = 0$ the spin and orbital angular momentum of the electron are measured along the z-axis and are found to be $+\hbar/2$ and 0 respectively. **Find the expectation value of $\hat{S}_x$ and $\hat{S}_z$ as functions of time.**

   Hint: You will need to switch back and forth between the coupled and uncoupled bases.