PH 500 Problem Set #5

So far we have focused on two-state systems — systems where the states were two-component vectors. This was just convenient for calculations; none of the basic concepts actually depended on this limitation. Now we expand the situation slightly and consider higher–dimensional cases. Again, the things I want you to do are in **bold**.

1. Suppose we have *two* spins. Then our state has four components. This four is not $2+2$, it's 2×2 (so for three spins, there are *eight* states). The states can be written as

   $$|\psi\rangle = a|+\rangle|+\rangle + b|+\rangle|-\rangle + c|-\rangle|+\rangle + d|-\rangle|-\rangle \tag{1}$$

   where each basis element is now made up of an eigenstate of $\hat{\sigma}_z$ for the first spin and an eigenstate of $\hat{\sigma}_z$ for the second spin. For a normalized state, the coefficients satisfy $|a|^2+|b|^2+|c|^2+|d|^2=1$. Now we have angular momentum operators $\hat{S}_{1x}$, $\hat{S}_{2x}$, $\hat{S}_{1y}$, $\hat{S}_{2y}$, $\hat{S}_{1z}$, and $\hat{S}_{2z}$. Each operator acts (just like before) on the corresponding spin, and leaves the other spin alone. And all the operators for the first spin commute with all the operators for the second spin.

   We can imagine putting these two spins into a magnetic field as before, without considering any interactions between them. The Hamiltonian is then

   $$\hat{\mathcal{H}} = -\gamma(\hat{\boldsymbol{S}}_{\mathbf{1}} + \hat{\boldsymbol{S}}_{\mathbf{2}})\cdot\boldsymbol{B} = -\gamma\frac{\hbar}{2}(\hat{\boldsymbol{\sigma}}_{\mathbf{1}} + \hat{\boldsymbol{\sigma}}_{\mathbf{2}})\cdot\boldsymbol{B} \tag{2}$$

   where $\hat{\boldsymbol{\sigma}}_{\mathbf{1}}$ acts on the first spin and $\hat{\boldsymbol{\sigma}}_{\mathbf{2}}$ acts on the second spin. **For $\boldsymbol{B}=B_0\hat{\boldsymbol{z}}$, Write this Hamiltonian as 4×4 matrix in the basis given above.**

   Since the spins don't interact with each other, the first spin and the second spin can be treated separately. So we have just written down two copies of the same problem. We could start each spin in a different state and find its time evolution using the results from the previous problem set.

   Next we'll let the spins interact. For example, suppose we consider a new possible term in the Hamiltonian:

   $$\hat{\mathcal{H}}' = -g'\hat{\boldsymbol{S}}_{\mathbf{1}}\cdot\hat{\boldsymbol{S}}_{\mathbf{2}} = -g'\left(\hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}\right) \tag{3}$$

   This describes an interaction in which the two spins want to align with each other (which happens in real spin systems).

   We still have a 4 by 4 Hamiltonian, which in principle we could just diagonalize to find the energy eigenstates. But it will be helpful to develop some techniques to make this job simpler, especially since we will eventually want to be able to deal with large (or even inifinte) dimensional matrices.

   Find the result of acting with $\hat{\boldsymbol{\sigma}}_{\mathbf{1}}\cdot\hat{\boldsymbol{\sigma}}_{\mathbf{2}}$ on each of the basis vectors above. Hint: This operator is the sum of three terms (for x, y and z). In each term, the $\hat{\sigma}_1$ operator acts on the first spin and the corresponding $\hat{\sigma}_2$ operator acts on the second spin. Then you can write the result as a linear combination of the original basis vectors. **Write the matrix for $\hat{\mathcal{H}}'$ in this basis.**

2. When we have new operators, we should always worry about their commutators. We found that the different components of $\hat{\boldsymbol{S}}$ don't commute, so they can't all be diagonalized at once. We could diagonalize $\hat{S}_z$ and $\hat{S}^2$ at the same time since they do commute. Now, all of the $\hat{\boldsymbol{S}}_{\mathbf{1}}$ operators commute with all of the $\hat{\boldsymbol{S}}_{\mathbf{2}}$ operators, since they act on different spins. So we can diagonalize $\hat{S}_{1z}$, $\hat{S}_{2z}$, $\hat{S}_1^2$, and $\hat{S}_2^2$ at once. But there's another choice that's also convenient. Define the *total spin* $\hat{\boldsymbol{J}} = \hat{\boldsymbol{S}}_{\mathbf{1}} + \hat{\boldsymbol{S}}_{\mathbf{2}}$. **Show that $\hat{J}^2$ commutes with $\hat{J}_z$, $\hat{S}_1^2$, and $\hat{S}_2^2$.** Hint: Use $(\hat{\boldsymbol{S}}_{\mathbf{1}} + \hat{\boldsymbol{S}}_{\mathbf{2}})^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\boldsymbol{S}}_{\mathbf{1}} \cdot \hat{\boldsymbol{S}}_{\mathbf{2}}$.

3. **Find the simultaneous eigenstates of $\hat{J}^2$ and J_z (that is the set of four vectors that are eigenstates of both operators). Find the corresponding eigenvalues of each operator.** Hint: To help you check your answer, I will try to give you some physical insight into what is going on. What we are doing is *adding the angular momentum* of the two spins. In quantum mechanics, we don't have an operator for the magnitude of the angular momentum. The best we can do is to look at $\hat{S}^2$, which is the magnitude squared. In the previous problem set you should have found its eigenvalue was $\hbar^2\frac{3}{4}$. In general, any angular momentum operator J^2 has eigenvalues $\hbar^2 j(j+1)$, where j is an integer or a half–integer. We can think of $\hbar j$ as the magnitude of the angular momentum (which is *quantized*). We would have expected this would yield an eigenvalue $\hbar^2 j^2$, but there is another $\hbar^2 j$ — roughly, to leave "room for uncertainty." The two-state system we have been considering is then just the case of $j = \frac{1}{2}$.

   If we add two angular momenta j_1 and j_2, we have to ask whether they add or cancel out (since maybe the angular momenta are in the same direction, and maybe they are in the opposite direction, and maybe they are somewhere in between). The answer is that we get all the (quantized) possibilities: the magnitude of the resulting angular momentum can range from $|j_1 - j_2|$ to $(j_1 + j_2)$, and *all the integer steps in between.* In the case we are looking at, we are adding angular momenta with $j_1 = \frac{1}{2}$ and $j_2 = \frac{1}{2}$. We're not going to derive the general result (which would require us to develop the whole theory of angular momentum) at this point, but the answers you get for this problem should be consistent with the result you would get from this argument.

4. Continuing with the two–spin problem, define the raising and lowering operators
$$\hat{J}^{\pm} = \hat{J}_x \pm i\hat{J}_y \tag{4}$$
   Denote the four states you found in the previous problem as $|j\ j_z\rangle$, where $\hbar^2 j(j+1)$ is the eigenvalue of $\hat{J}^2$ and $\hbar j_z$ is the eigenvalue of $\hat{J}_z$. **Find the action of both of these operators on each of these four states.** What is being "raised" and "lowered" by these operators?

5. Consider two spins subject to the Hamiltonian
$$\hat{\mathcal{H}} = -\alpha \hat{\boldsymbol{S}}_{\mathbf{1}} \cdot \hat{\boldsymbol{S}}_{\mathbf{2}} \tag{5}$$
   At time $t = 0$ we measure the z component of angular momentum of each spin separately and find $+\dfrac{\hbar}{2}$ for the first spin and $-\dfrac{\hbar}{2}$ for the second spin. **Find the expectation values of $\hat{S}_{1x}$, $\hat{S}_{2x}$, S_{1y}, $\hat{S}_{2y}$, $\hat{S}_{1z}$, and $\hat{S}_{2z}$ as functions of time.**