PH500 Problem Set #2

This problem set is a little more abstract than the previous one; we're establishing a lot of the formalism we will use in the next problem sets to do more concrete calculations. Although it looks long, some of the problems are quick once you get through all the definitions. Again, the things I want you to do are in **bold**.

1. We've gotten familiar with $\langle\phi|\psi\rangle$, which we called the inner product. It is a scalar; if $|\phi\rangle$ is a unit vector, it gives the component of $|\psi\rangle$ in the direction of $|\phi\rangle$.

   We can also define the *outer product* of two vectors. For

$$\begin{aligned} |\psi\rangle &= \begin{pmatrix} a \\ b \end{pmatrix} \\ |\phi\rangle &= \begin{pmatrix} c \\ d \end{pmatrix} \end{aligned} \tag{1}$$

   we have

$$|\psi\rangle\langle\phi| = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c^* & d^* \end{pmatrix} = \begin{pmatrix} ac^* & ad^* \\ bc^* & bd^* \end{pmatrix} \tag{2}$$

   which is a *matrix*.

   The most useful outer product is that of a normalized vector with itself,

$$\hat{P}_\phi = |\phi\rangle\langle\phi| \tag{3}$$

   which is called a *projection operator*. What does this operator do? Well, imagine operating on another vector $|\psi\rangle$. First, it takes the component of $|\psi\rangle$ in the direction of $|\phi\rangle$, and then it multiplies this number by the vector $|\phi\rangle$. So it has "projected out" the component of $|\psi\rangle$ parallel to the unit vector $|\phi\rangle$.

   (a) **Show that $\hat{P}_\phi^2 = \hat{P}_\phi$. Find the possible eigenvalues of $\hat{P}_\phi$. Explain how this result is consistent with $\hat{P}_\phi$ being a projection operator, and describe the corresponding eigenvectors.**

   (b) Suppose we have an orthonormal basis $\{|e_j\rangle\}$. **Show that**

$$\sum_j |e_j\rangle\langle e_j| = \hat{1} \tag{4}$$

   where $\hat{1}$ is the identity operator, $\hat{1}|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle$. **Explain this result in words.**

   (c) **Check the previous result explicitly for the basis of eigenvectors of $\hat{\sigma}_x$ you computed in the previous problem set.**

2. On the previous problem set, we introduced the expectation value of an operator in a given state. Since we will use it so much in the rest of this problem set, I will introduce a compact notation

$$\langle\psi|\hat{A}|\psi\rangle \equiv \langle\hat{A}\rangle_\psi \tag{5}$$

In the last problem set we found this was the mean value of the results of the measurement, weighted by their probability. Next, we will investigate the *deviation* from this mean. For a given operator $\hat{A}$, consider the operator

$$\widehat{\Delta A} = \hat{A} - \langle\hat{A}\rangle \tag{6}$$

What does this mean? For a given operator $\hat{A}$, I've defined a new operator called $\widehat{\Delta A}$. Here $\langle\hat{A}\rangle$ is just a *number* — the expectation value of $\hat{A}$ in whatever state this operator is acting on.

So if I act with this operator on a state $|\psi\rangle$, what I get is the vector $\hat{A}|\psi\rangle$ minus the number $\langle\hat{A}\rangle_\psi$ times the state $|\psi\rangle$.

(a) Show that $\langle\widehat{\Delta A}\rangle_\psi = 0$ in any state $|\psi\rangle$.

(b) Show that if $|\psi\rangle$ is an eigenstate of $\hat{A}$, then $\widehat{\Delta A}|\psi\rangle = 0$.

3. Our goal was to have $\widehat{\Delta A}$ represent the deviation of $\hat{A}$ from its average value. Since $\hat{A}$ deviates as much above as below, $\widehat{\Delta A}$ always averages to zero (as you found in the last problem). So we can't get any more information from its expectation value — which is the usual situation in statistics. The solution is to define the *variance*, which is the *square* of the deviation. By averging the variance — which is always positive — and then taking square root of the result, we obtain the *standard deviation*, which represents the average magnitude of the deviation from the mean.

Let's repeat that in equations. Define

$$(\widehat{\Delta A})^2 = \left(\hat{A} - \langle\hat{A}\rangle\right)^2 \tag{7}$$

Then the *uncertainty in $\hat{A}$ in the state $|\psi\rangle$* is given by

$$\Delta_{\hat{A},\psi} = \sqrt{\langle(\widehat{\Delta A})^2\rangle_\psi} \tag{8}$$

In other words, take the *operator* $\widehat{\Delta A}$, *square* it, take the *expectation value* in the state $|\psi\rangle$ (which is always a positive number), and then take the square root of that number.

Show that

$$\Delta_{\hat{A},\psi}{}^2 = \langle\hat{A}^2\rangle_\psi - \langle\hat{A}\rangle_\psi^2 \tag{9}$$

4. Prove the *Schwartz inequality*: **For any two states $|\phi\rangle$ and $|\psi\rangle$, show that**

$$|\langle\phi|\psi\rangle| \leq \sqrt{\langle\phi|\phi\rangle\langle\psi|\psi\rangle} \tag{10}$$

Hint: define

$$|\gamma\rangle = |\phi\rangle - \frac{|\psi\rangle\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle} \tag{11}$$

and consider $\langle\gamma|\gamma\rangle$. **Explain why this makes sense geometrically. When does the inequality become an equality?**

5. Now let's put it all together to prove the *uncertainty principle*.

Starting with a state $|\psi\rangle$ and two Hermitian operators $\hat{A}$ and $\hat{B}$, define

$$|f\rangle = (\hat{A} - \langle A\rangle_\psi)|\psi\rangle \tag{12}$$

and

$$|g\rangle = (\hat{B} - \langle B\rangle_\psi)|\psi\rangle \tag{13}$$

Show that

$$\Delta_{\hat{A},\psi}{}^2\Delta_{\hat{B},\psi}{}^2 \geq |\langle f|g\rangle|^2 \tag{14}$$

Hint: use the Schwartz inequality.

Now the norm squared of any complex number z is just its real part squared plus its imaginary part squared, so we have

$$|z|^2 = (\Re z)^2 + (\Im z)^2 \geq (\Im z)^2 = \left[\frac{1}{2i}(z - z^*)\right]^2 \tag{15}$$

Choosing $z = \langle f|g\rangle$, we have

$$\Delta_{\hat{A},\psi}{}^2\Delta_{\hat{B},\psi}{}^2 \geq \left[\frac{1}{2i}(\langle f|g\rangle - \langle g|f\rangle)\right]^2 \tag{16}$$

Show that

$$\langle f|g\rangle = \langle\hat{A}\hat{B}\rangle_\psi - \langle\hat{A}\rangle_\psi\langle\hat{B}\rangle_\psi \tag{17}$$

Similarly

$$\langle g|f\rangle = \langle\hat{B}\hat{A}\rangle_\psi - \langle\hat{B}\rangle_\psi\langle\hat{A}\rangle_\psi \tag{18}$$

(note the crucial difference in the order!) and we have found that

$$\Delta_{\hat{A},\psi}{}^2\Delta_{\hat{B},\psi}{}^2 \geq \left(\langle\frac{1}{2i}[\hat{A},\hat{B}]\rangle_\psi\right)^2 \tag{19}$$

where $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the *commutator* of $\hat{A}$ and $\hat{B}$. This is the *uncertainty principle*. Note that the right–hand side has an i^2, which makes it look negative. But that's not the case; remember that, as you showed in the last problem set, $[\hat{A},\hat{B}]/(2i)$ is Hermitian. A Hermitian operator has real eigenvalues, and a Hermitian operator *squared* has all positive (or zero) eigenvalues. So really the right–hand side, like the left–hand side, is a positive–definite expression.

6. **For the state**

$$|\psi\rangle = \frac{1}{\sqrt{3}}|+\rangle + i\sqrt{\frac{2}{3}}|-\rangle \tag{20}$$

you studied in the previous problem set, verify the uncertainty principle (by computing both sides) for

(a) $\hat{A} = \hat{\sigma}_x \quad \hat{B} = \hat{\sigma}_y$

(b) $\hat{A} = \hat{\sigma}_y \quad \hat{B} = \hat{\sigma}_z$

(c) $\hat{A} = \hat{\sigma}_z \quad \hat{B} = \hat{\sigma}_x$

7. While we're discussing commutators, one more result will be very useful to know. Suppose $\hat{A}$ and $\hat{B}$ *commute*, so $[\hat{A}, \hat{B}] = 0$. Suppose that you find an eigenvector $|\lambda_i\rangle$ of $\hat{A}$ with eigenvalue λ_i, and suppose that it is nondegenerate, that is, it is the *only* eigenvector with this eigenvalue. **Show that $|\lambda_i\rangle$ is also an eigenvector of $\hat{B}$.** This relatively simple result has an important consequence: *If $\hat{A}$ and $\hat{B}$ commute, we can construct a basis $\{ \ |\lambda_i\rangle \ \}$ that are eigenvectors of both operators.* For nondegenerate eigenvectors, you have already shown this result. For degenerate eigenvectors, any linear combination of eigenvectors with the same eigenvalue is still an eigenvector. So another exercise (which I'm not asking you to do) shows that you can *choose* combinations that are eigenvectors of both operators.