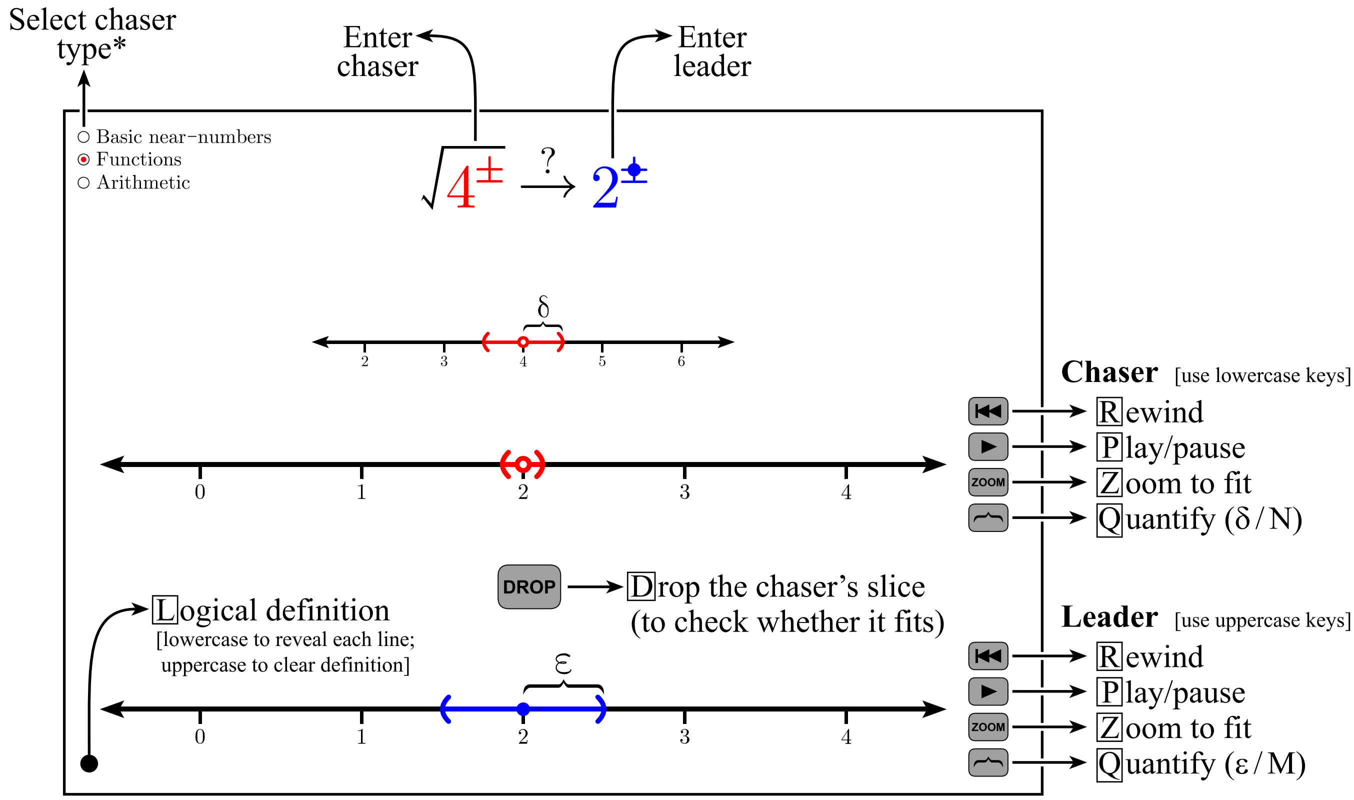
Please note that some browsers require the animation to be clicked first to activate it.



* Enter the leading and chasing near-numbers as in the relevant animations

After entering a chaser and leader, the question of whether the arrow relation holds can be explored visually, as follows: each basic near-number provides a specific model of convergence or divergence, painting ever more refined targets of some ideal location on the line. The question of whether some other near-number fits this model can be considered as a game of follow-the-leader: in order that $\alpha \longrightarrow \beta$, the chaser ($\alpha$) must fit into *every* target painted by the leader (β). This can be visually explored as follows:

① Let the leader squeeze as close as you like toward its ideal, then pause it.
[i.e., Let $\varepsilon > 0$ be given]

② Set the chaser going and see if it can fit into the target painted by the leader; the chaser can be "dropped" at any time to check whether it fits into the target—if so, so far so good.
[i.e., Is there a corresponding $\delta > 0$?]

If the chaser can fit into every target painted by the leader, then it "fits" the model, and the arrow relation holds—if not, then the arrow relation does not hold.

Note that the process of arrow-evaluating a near-number expression involves two distinct steps: first, analyzing the expression and determining (through the expression or its animation) what its behavior should be. Second, once its behavior has been determined, enter the relevant leader to verify that it does, indeed, fit that model.

Quantifying the definition: a stepwise approach

- **Step 1:** Quantifying the basic near-numbers to obtain set conditions

  Before exploring the logical definition of the arrow relation (or the limit), it is important to first establish exactly how we quantify each basic near-number in terms of some variable (e.g., ε or M) and how for each value of that variable, the near-number gives some condition specifying a location on the real line. For example:

  - For each $\varepsilon > 0$, $L^{\dot{\pm}}$ gives us the the condition "$L - \varepsilon < \# < L + \varepsilon$".
  - For each $\varepsilon > 0$, $a^{\pm}$ gives us the the condition "$a - \varepsilon < \# < a + \varepsilon$ and $\# \neq a$".
  - For each $M > 0$, $\pm\infty$ gives us the condition "$|\#| > M$".
  - For each $N > 0$, $+\infty$ gives us the condition "$\# > N$".

  There is no need for rote memorization—these are easily recovered from the quantified pictures!

- **Step 2:** The arrow relation for basic near-numbers

  The proper view of the relation $\alpha \to \beta$ is to consider β to provide a model of behavior, into which α either fits ($\to$) or does not fit ($\not\to$). Viewing the β as the "leader" and α as the "chaser" thus gives us:

  - α can fit into every target painted by β.

  More specifically, realizing that we must specify the target before we try to hit it:

  - For each target painted by β, if we let α squeeze enough, it will fit within the target.

  This notion can be explored visually before adding the quantifiers, in order to see that it is what we mean to say. Once so convinced, we simply name the quantifying variables for the leader and chaser:

  - For each [quantifier of the leader, β], there is some [quantifier of the chaser, α] so that the chosen slice of the chaser, α, fits within the given target slice of the leader, β.

  Finally, for one slice to "fit within" another simply means that every number in the first is also in the second, which gives our general template for the logical definition:

$\alpha \to \beta$	means:	For each [quantifier of β], there is some [quantifier of α] so that for all x with [condition for α], x has [condition for β].

  From this, we can now write the fully quantified logical definition for any given case. For example:

  - $2^{+} \to 2^{\dot{\pm}}$ means: For each $\varepsilon > 0$, there is some $\delta > 0$ so that for all x with $2 < x < 2 + \varepsilon$, we have $2 - \varepsilon < x < 2 + \varepsilon$.
  - $+\infty \to \pm\infty$ means: For each $M > 0$, there is some $N > 0$ so that for all x with $x > N$, we have $|x| > M$.

  Note that the logical mechanics of proving such a statement can now be explored without interference from any complicated functions or arithmetic!

- **Step 3:** Functions

  The only difference between the statement $\alpha \to \beta$ and the statement $f(\alpha) \to \beta$, as demonstrated in the animation, is that the numbers in each slice of $f(\alpha)$ are the results of applying f to each number x in the corresponding slice of α—so we check that each $f(x)$ is in the target slice:

$f(\alpha) \to \beta$	means:	For each [quantifier of β], there is some [quantifier of α] so that for all x with [condition for α], $\underline{f(x)}$ has [condition for β].

  The graph of the function need not (and should not) complicate things by entering into this definition—all that should be considered is the manner in which the function operates on the slices of α.

Quantifying the definition (continued)

- **Step 4:** Near-number arithmetic

  The definition of the arrow relation $\alpha * \beta \to \gamma$ when $*$ is a binary operation follows as illustrated in the animation: for each target slice of the leader, γ, some slice of the chaser, $\alpha * \beta$, must fit within it. As usual, the slices of $\alpha * \beta$ consist of the numbers $x * y$, where x and y are numbers in the corresponding slices of α and β. Thus, the definition quantifies as usual, checking that $x * y$ fits within the target slice:

$\alpha * \beta \to \gamma$	means:	For each [quantifier of γ], there is some [quantifier of α] and some [quantifier of β] so that for all x with [condition for α] and y with [condition for β], $x * y$ has [condition for γ].

  It should be noted that we can thus explore the definition of the arrow relation for arithmetic operations (and prove them) without reference to any particular function (rather than the more cumbersome usual treatment of the case of $f(x) + g(x)$, which unnecessarily complicates the operation of addition by introducing two functions as well).

Other properties of the arrow relation

- It is easily seen (and proven) that the arrow relation on near-numbers is both *reflexive* and *transitive*:
  - for any near-number α, $\alpha \to \alpha$;
  - and for any α, β, γ, if $\alpha \to \beta$ and $\beta \to \gamma$, then $\alpha \to \gamma$.
- As its symbol suggests, the arrow relation is *not symmetric*.
  - We need look no farther than the basic near-numbers to see this: $0^{+} \to 0^{\pm}$, but $0^{\pm} \not\to 0^{+}$.
  - This is particularly important for indeterminate forms; if $\alpha \to \bigstar$, α could be anything, just as when $\alpha \to L^{\dot\pm}$, it's entirely possible that $\alpha \to L^{\dot+}$ or any of the other basic finite near-numbers squeezing to L.
  - Uniqueness of limits is an immediate consequence of the fact that if $L \neq K$, then no nonempty α fits into both $L^{\dot\pm}$ and $K^{\dot\pm}$.