ELEMENTARY TOPOLOGY

Notes for MA 432

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I. What is it All About?

"topology [*topo*- (from Greek *topos* place + -*logy*] 1. Topographical study of a particular place; specif., the history of a region as indicated by its topography. 2. *Anat*. The anatomy of a particular region of the body. 3. *Math*. The doctrine of those properties of a figure unaffected by any deformation without tearing or joining."

- Webster's New Collegiate Dictionary

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"Modern mathematics rests on the substructure of **mathematical logic** and the **theory of sets**... Upon this base rise the two pillars that support the whole edifice: **general algebra** and **general topology**."

- Lucienne Felix

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"Point set topology is a disease from which later generations will regard themselves as having recovered."

- Henri Poincare

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"The concept of convergence is fundamental for analysis and makes its appearance in different situations. Hence, for example, one considers convergent sequences of numbers, or more generally, of points lying in a Euclidean space and convergent sequences of functions. Here even different concepts of convergence may be used such as ordinary convergence, uniform convergence or mean convergence. The concept of continuity is closely connected with that of convergence. A real function is continuous if and only if the function applied to every convergent sequence lying in its domain of definition transforms such a sequence to a sequence which is also convergent. We shall be concerned in what follows with exhibiting in full generality those concepts which are connected with convergence and continuity. Thus we proceed axiomatically: we consider sets and endeavour to define structures on them so that it will be possible to speak of continuous mappings and of convergence. The elements of such sets will be called points without thereby attaching any fixed significance to this terminology. It is possible that the 'points' of such a set are functions defined on another set or some other mathematical object."

- Horst Schubert

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"It is tempting to call topology **rubber-sheet geometry** and to hope that the manufacturers of two-way-stretch foundation garments will subsidize a chair for the study of this branch of mathematics. But we shall see that more general transformations than those afforded by stretching rubber sheets must be studied, and manufacturers of feminine underwear do not seem to need the help of the higher mathematics in their study of foundations."

- Dan Pedoe

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"If anyone asks, 'What is topology?' the most correct answer is 'Topology nowadays is a fundamental branch of mathematics and like most fundamental branches of mathematics does not admit of a simple concise definition.' Topologists are indeed investigating widely different problems and are using a multitude of techniques. Topology is today one of the most rapidly expanding areas of mathematical thought.... Topology is valuable in its own right in so far as any well-developed mathematical study is valuable, or in fact, any aesthetically pleasing creation of the human mind is valuable."

- Michael Gemignani

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"A topologist is a person who doesn't know the difference between a doughnut and a coffee cup."

- Anonymous

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"Die Topologie hat es mit solchen Eigenschaften geometrischer Figuren zu tun, die bei topologischen Abbildiungen, d.h. umkehrbar eindeutigen und umkehrbar stetigen Abbildungen ungeändert bleiben."

- H. Seifert and W. Threlfall

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"Today the angel Topology and the demon Abstract Algebra struggle for the soul of each of the mathematical domains."

- Herman Weyl

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"Unacountably, Nerzhin, the younger of the two men, the failure with no academic title, asked the questions, the creases around his mouth sharply drawn. And the older man answered as if he were ashamed of his unpretentious personal history as a scientist: evacuation in wartime, re-evacuation, three years of work with K-—-, a doctoral dissertation in mathematical topology. Nerzhin, who had become inattentive to the point of discourtesy, did not even ask Verenyov the subject of his dissertation in that dry science in which he himself had once taken a course. He was suddenly sorry for Verenyov. Quantities solved, quantities not solved, quantities unknown–topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of some use to someone, but as for now ..."

Aleksandr I. Solzhenitsyn
The First Circle

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"It frequently happens that when getting a cup of coffee one forgets the cream. The trick, here, is not to go and get the cream, but to take the cup to it. The first way involves four trips: going for the cream, bringing it to the table, taking it back right away, and returning to the coffee. The other way involves two: taking the cup to the refrigerator and returning with the cup. This cannot be helpfully expressed geometrically, but the kind of sequential planning used, though arithmetical, belongs rather in topology.

"In the use of electronic computers it is called Programming, and set-theoretic topology is its basis. In designing the fearsomely complicated circuits of these computers they make use of topological network analysis. Topology has found a place in astronomy, also, and indeed many other endeavors where mathematics is needed. These subjects are hardly everyday, but we use topological methods in many everyday acts, though unconsciously. Most descriptions of where something is are topological: The coat is in your closet; The school is the fourth house beyond the intersection of this street and Route 32; The Pen of my Aunt is in the Garden....

"The Renaissance marked several changes in scientific thinking and method, one of which is best exemplified by chemistry. Medieval alchemy concentrated on difference in kind –- difference in degree seemed less of the essence to them, and their chemistry never got off the ground. With the new ways of thinking, chemistry turned away from the qualitative to the quantitative –- from

Kind to Degree –- and they began to get order from chaos. Mathematics, on the other hand, had always leaned toward the quantitative method –- until topology, and the process seems to be reversed.

"But not really: in looking back ... we can see that while form and measurement are temporarily abandoned they crop up again in a more sophisticated guise, for quality has a quantity; kind has a degree, even if it is not measured with a yardstick. As Stephen Vincent Benet said (he was talking of Lincoln), yardsticks are good for measuring –- if you have yards to measure."

- Stephen Barr

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"Topology is a relatively new branch of mathematics. Those who took training in mathematics 30 years ago did not have the opportunity to take a course in topology at many schools. Others had the opportunity, but passed it by, thinking topology was one of those 'new fangled' things that was not here to stay. In that respect, it was like the automobile.

"One who is introduced to topology through popular lectures and entertaining articles may get the impression that topology is recreational mathematics. If he were to take a course in topology, expecting it to consist of cutting out pretty figures and stretching rubber sheets, he would be in for a rude awakening. If he pursued the subject further, however, he might be delighted to find that it is rich in substance and beauty."

- R. H. Bing

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"One can prove, using a modern branch of mathematics known as *topology*, that *any* map of the surface of the entire earth must have the same weakness; one cannot concoct a way of depicting the whole earth on a sheet of paper so that nearby points on the earth will always be close to each other on the map."

- Robert Osserman

Poetry of the Universe

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"It is perhaps the area of mathematics in which the greatest number of entirely new ideas has appeared, many of which have had unexpected repercussions in theories which seem very remote from it. It forms an imposing edifice, constantly under renovation, and of such complexity that very few specialists are capable of encompassing all of it."

- Jean Dieudonné

Mathematics - The Music of Reason

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"According to Woody Allen, fake rubber inkblots were originally 11 feet in diameter and fooled nobody. Later, however, a Swiss physicist `proved that an object of a particular size could be reduced in size simply by *making it smaller*, a discovery that revolutionized the fake inkblot business.´ This little tale could be interpreted as a parody of topology, a subject whose insights at first look do seem a little obvious....There is much more to topology than fake rubber inkblots.

- John Allen Paulos

Beyond Numeracy: Ruminations of a Numbers Man

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"It now lately sometimes seemed like a kind of black miracle to me," says Hal, "that people could actually care deeply about a subject or pursuit, and could go on caring this way for years on end. Could dedicate their entire lives to it. It seemed admirable and at the same time pathetic. We are all dying to give our lives away to something, maybe. God or Satan, politics or grammar, topology or philately - the object seemed incidental to this will to give oneself away, utterly."

-David Foster Wallace, Infinite Jest

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topology: it's the weirdest, wildest area of math

- Bryan Clair

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When he was a teenager the rigid drills of schooling had made him think that mathematics was just felicity with a particular kind of minutiae, *knowing things*, a sort of high-grade coin collecting. You learned relations and theorems and put them together.

Only slowly did he glimpse the soaring structures above each discipline. Great spans joined the vistas of topology to the infinitesimal intricacies of differentials, or the plodding styles of number theory to the shifting sands of group analysis. Only then did he see mathematics as a landscape, a territory of the mind to rove and scout.

(-Gregory Benford, in his novel *Foundation's Fear*, writing about the character Hari Seldon*)*

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A child[’s] ... first geometrical discoveries are topological ... If you ask him to copy a square or a triangle, he draws a closed circle.

– Jean Piaget

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This diagram [the mobius strip] can be considered the basis of a sort of essential inscription at the origin, in the knot which constitutes the subject. This goes much further than you may think at first, because you can search for the sort of surface able to receive such inscriptions. You can perhaps see that the sphere, that old symbol for totality, is unsuitable. A torus, a Klein bottle, a cross-cut surface, are able to receive such a cut. And this diversity is very important as it explains many things about the structure of mental disease. If one can symbolize the subject by this fundamental cut, in the same way one can show that a cut on a torus corresponds to the neurotic subject, and on a cross-cut surface to another sort of mental disease.

– Jacques Lacan

II. Notes on the Style

"Math 55l, Elementary Topology. Purpose: to give an introduction to the ideas, problems, and methods of point set topology. This course is a good preparation, but not an essential prerequisite, for a graduate course in topology. This course is also useful as background for analysis courses. Usually this course is based on students' presentations of their own proofs of theorems."

- Undergraduate Mathematics Course Outlines,
1970-71, University of Wisconsin

What is demanded of the student in this course is perhaps best described by Lucille S. Whyburn in her article "Student Oriented Teaching–The Moore Method" (*American Mathematical Monthly*, April 1970, pp.351-359): Our assumption is that
"the student has attained a certain stage of mathematical maturity, that he is interested in 'doing' mathematics, not in ferreting out what former mathematicians have done through reading of theorems and proofs or the application of knowledge thus gained to problems. Passive listening followed by exhibition on quizzes and examinations that he has understood what was said is not sufficient in this class; each member must want a piece of the action. The discovery of proofs and definition of concepts unknown to the student... challenge the student to exercise his own honor system about not reading related material or cribbing ideas from any

source."

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"That student is taught best who is told the least."

- R. L. Moore
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These notes contain the statements of various theorems, problems and questions. Your job is to supply, without consultation or reference to the mathematical literature, proofs for the theorems (or counter-examples if they are false), solutions of the problems and answers to the questions. At the beginning of each class period you will turn in a slip of paper with your name and the numbers of the theorems, problems and questions you have been able to solve. You will be called on to present your work before the class.

Many, perhaps most, of you do not as yet really know how to construct a complete and correct mathematical proof. Your previous mathematics courses probably did not stress this aspect. Learning this process will be a valuable part of this course for you.

You will fairly quickly discover that some of the theorems are easy to prove, while others are quite difficult. You should not necessarily try to prove all of the theorems. but rather, be willing to devote a good deal of time to those which intrigue you.

As to what you may expect from the teacher, I will quote from a note "To the Instructor" in a recent text by Philip Nanzetta and George Strecker which contains only definitions, examples and statements of theorems (no proofs): "'Teaching' from this book for the first time is likely to be a memorable experience. You forget just how difficult it is to quietly sit and watch someone present a proof different from the one you have in mind, or one that is too sketchy or is burdensomely detailed. But this pain is ultimately worth it. The reward of actually seeing a student who didn't even know what a proof was at the beginning present a beautiful, polished proof after some months of work justifies the pain on your part and the effort on his. Patience is called for, and criticism, and trick questions, and traps. Blind alleys must be followed to the end. But most of the time you must sit and be quiet."

III. The Heart of the Matter

The objects which we will be studying in this course are "topological spaces." Before we discuss topological spaces it is necessary that we first accumulate some facts about sets. Since it is impossible to define all words, we will assume that we know what it means for something to be a set and what it means for something to be an element of a set. The words "set" and "collection" will be used synonymously.

Notation: If **A** is a set, then $\mathbf{x} \in \mathbf{A}$ means that **x** *is an element of* **A**, or equivalently, **x** *is a member of* **A**, or **x** *belongs to* **A** or **x** *is in* **A.** The set with no elements is called the **empty set** and is represented by the symbol **Ø.**

Definition: Two sets are **equal** if and only if they have precisely the same elements.

Definitions: If **A** is a set, the statement that **B** is a **subset** of **A** means that **B** is a set, and that each element of **B** is an element of **A.** If **A** is a set the notation $\mathbf{B} \subset \mathbf{A}$ means that **B** is a subset of **A.**

If **A** is a set and **B** is a subset of **A**, then **B** is a **proper subset** of **A** if and only if there is an element of **A** which is not an element of **B.**

Definition: If each of **A** and **B** is a set, the **union** of **A** and **B,** denoted $\mathbf{A} \cup \mathbf{B}$, is the set **C** such that **x** is an element of **C** if and only if either **x** is an element of **A** or of **B.**

Definition: If **G** is a collection, each element of which is a set, the **union** of the sets of **G** is the set **X** such that **y** is an element of **X** if and only if there is an element **g** of **G** such that **y** is an element of **g**.

Definition: If each of **A** and **B** is a set, then the **intersection**, or **common part** of **A** and **B,** denoted **A** $\cap$**B,** is the set **C** such that **x** is an element of **C** if and only if **x** is an element of **A** and **x** is an element of **B.**

Definition: If **A** and **B** are sets and have no element in common, then **A** and **B** are **disjoint**. This is denoted **A** $\cap$**B** = **Ø.**

Definition: If **G** is a collection of sets, then the **intersection** or **common part** of the sets of **G** is the set **C** such that **x** is an element of **C** if and only if for each set **g** of **G**, **x** is an element of **g**.

Notation: Suppose that **P** is a "well defined property" that an object may or may not possess. We use the notation **{x**: **x** has property **P}** to denote the set of all objects with property **P**.

For example, **{x**: **x** is a real number and **x** > 0**}** is just the set of positive real numbers, **{x**: **x** is a student at Middlebury College**}** is just the collection of students at this college, and if **A** and **B** are sets, then **{x**: either **x** is in **A** or **x** is in B**}** is just the union of **A** and **B.**

We will assume the following axiom about the positive integers. Feel free to use it whenever appropriate.

Axiom of Mathematical Induction: Every non-empty set of positive integers has a smallest element; that is, if **A** is a set all of whose elements are positive integers and **A** is not the empty set, then there exists an element s in **A** such that $s \leq x$ for all x in **A**.

Definition: The statement that **F** is a **function** means that **F** is a collection of ordered pairs, such that no two of these pairs have the same first term.

Definitions: Suppose that **F** is a function. The **domain** of **F** is the set **X** such that **x** is an element of **X** if and only if **x** is the first term of some element of **F**. The **range** of **F** is the set **Y** such that **y** is an element of **Y** if and only if **y** is a second term of some element of **F**. If **x** is the first term of an element of **F**, then **F**(**x**), the **value of F at x**, denotes the second term of the ordered pair of **F** whose first term is **x** The function **F** is said to be a function from **X onto Y**. If **Z** is a set such that **Y** is a subset of **Z**, then **F** is a function from **X into Z**. The notation **F:X** $\rightarrow$ **Z** means that **F** is a function from **X** into **Z**. If **A** is a subset of **X,** then **F(A)** denotes the set of all elements **F**(a) where a is an element of **A.**

Example: Let **R** denote the set of all real numbers. Let **F** be **{**(**x**,**x**2): **x** is an element of **R }** . Then **F** is a function. The domain of **F** is **R**, the range of **F** is **{x**: **x** is in **R** and **x** $\geq$ 0**}**, **F**(2) = 4, and we have **F**:**R** $\rightarrow$ **R**

Definition: Suppose that **F** is a function. The statement that **F** is **one-to-one** means that no two elements of **F** have the same second term. In other words, if **x** and **y** are distinct elements of the domain of **F**, then **F**(x) is different from **F**(y). Note that in the above example, **F** is not one-to-one because **F**(2) = **F**(-2).

Question 1: Let **f** be a function from a set **X** into a set **Y** and let **A** and **B** be subsets of **X**. Which of the following statements are always true?

(a) **f(A $\cup$ B) = f(A) $\cup$ f(B)**

(b) **f(A $\cap$B) = f(A) $\cap$f(B)**

(c) **Y - f(A)** = **f(X - A)** for each subset **A** of **X**.

Note that if **C** and **D** are sets, then **C - D** denotes the set of all elements of **C** which are not members of **D**.

(d) If **f** is a one-to-one function and **g** is a subset of **f**, then **g** is a one-to-one function.

(e) Does statement (a) remain true if the union of two sets is replaced by the union of an arbitrary collection of subsets of **X**?

(f) Does statement (b) remain true if the intersection of two sets is replaced by the intersection of an arbitrary collection of subsets of **X**?

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Definition: Suppose that **X** and **Y** are sets. Then the statement that **X is equivalent to Y**, denoted **X ~ Y**, means that there is a one-to-one function from **X** onto **Y**.

Example: Let **F** = { (*n*, *n*+1): n is a non-negative integer}. Then **F** is a one-to-one function from the set of non-negative integers onto the set of positive integers.

Notation: Henceforth, we will let **J** denote the set of positive integers, **R** denote the set of real numbers, and **Q** denote the set of rational numbers.

Example: Let **F** be **{** (*x*, arctan *x*): *x* is in **R}** . Then **F** is a one-to-one function from **R** onto **{***x*: *x* is in **R** and $-\pi/2 < x < \pi/2$**}** .

Example: Let **G** be **{** (*x*,exp(*x*)): *x* is in **R}** where exp(*x*) is the usual exponential function. Then **G** is a one-to-one function from **R** onto **{***x*: *x* is in **R** and $x > 0$**}** .

Exercise 1: Show that if **A** is the set of positive integers and **B** is the set of even positve integers, then **A** ~ **B**

Exercise 2: Show that if **A** is the set of positive integers and **Z** is the set of integers, then **A ~ Z**.

Exercise 3: Show that **C** is the set {0, 1, 1/2, 1/3, 1/4,…} and **D** is the set {1, 1/,2, 1/3, ..}, then **C ~ D**

***Exercise 4*:** Let **A** be the closed unit interval; that is, **A** = {*x: x* is a real number and $0 \le x \le 1$}. Let **B** be the open unit interval; that is, **B** = {*x* : *x* is a real number and $0 < x < 1$ }. Is **A~ B**? Is **B ~ A** ?

Theorem 1: If **X** is a set, then **X ~ X**.

Theorem 2: Suppose **X** and **Y** are sets. If **X ~ Y**, then **Y ~X**.

Theorem 3: Suppose that **X, Y**, and **Z** are sets. If **X ~ Y** and **Y ~ Z**, then **X ~ Z**.

Definition: Suppose that **A** is a set. Then **A** is **infinite** if and only if **A** is equivalent to a proper subset of itself. A set **A** is **finite** if and only if it is not infinite.

Exercise 5: **J** is infinite

Exercise 6: **R** is infinite

Exercise 7: The empty set is finite.

Exercise 8: The set **A** = {1} is finite.

Definition: Suppose that **A** is a set. Then **A** is **countable** if and only if there is a subset **H** of **J** such that **A ~ H**.

Theorem 4: Suppose **A** and **B** are sets. If **A ~ B** and **A** is infinite, then **B** is infinite.

Theorem 5: Suppose that **A** and **B** are sets. If **A** is a subset of **B** and if **A** is infinite, then **B** is infinite.

Theorem 6: Suppose that **A** is a set, **B** is a subset of **A**, **C** is a subset of **B** and **A ~ C**. Then **A ~ B.**

Theorem 7: Suppose that **A** is a set, **M** is a set, **B** is a subset of **A**, **N** is a subset of **M**, **A ~ N** and **M ~ B.** Then **A ~ M**.

Theorem 8: If **A** is a countable set and **B** is a countable set, then the union **A** $\cup$**B** is a countable set.

Theorem 9: If **G** is a countable collection of sets, each of which is countable, then the union of the sets of **G** is countable.

Notation: If **A** is a set, we will let **2A** denote the collection of all subsets of **A.**

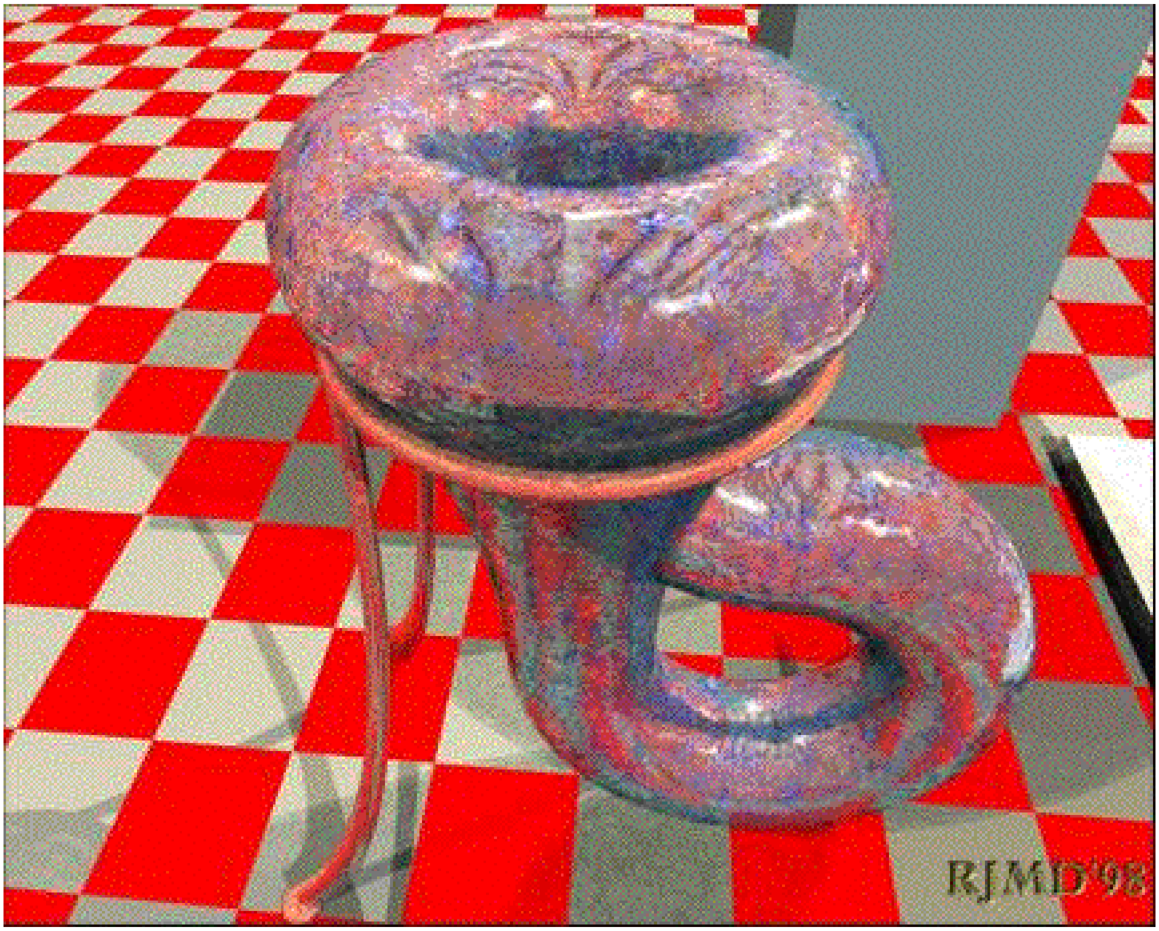
Question 2: Does there exist a set **A** such that **A ~ 2A**?

Definitions: Suppose **a** and **b** are real numbers and that **a** < **b**. The **closed interval** from **a** to **b,** denoted **[a,b]** is **{x**: **x** is a real number and **a** ≤ **x** and **x** ≤ **b}** while the **open interval** from **a** to **b**, denoted **(a,b)** is the set **{x**: **x** is a real number, **a** < **x** and **x** < **b}** .

We will assume the following axiom about the real numbers. Feel free to use it whenever appropriate.

Axiom: The intersection of any countable nested collection of closed intervals of real numbers is nonempty. By a **nested collection,** we mean one in which the *i+1* st interval is a subset of the *i* th interval for each *i* = 1,2,3,...

Theorem 10: Suppose that **X** is a non-empty finite set. Then there is a positive integer **n** such that **X** is equivalent to **{ z: z** is an element of **J** and **z** ≤ **n}** .



We are now ready to begin our discussion of topological spaces.

Definition. Suppose that **X** is a set. The statement that **T** is a **topology** for **X** means that **T** is a collection of subsets of **X** such that:

(i) **X** is an element of **T** and **Ø** is an element of **T**;

(ii) The union of the sets of any subcollection of **T** is an element of **T**; that is, if **S** is a subset of **T** then $\cup$ **{ y**: **y** in **S}** is an element of **T**, and

(iii) If **U** is an element of **T** and **V** is an element of **T**, then **U** $\cap$**V** is an element of **T**.

Definition. The statement that **S** is a **topological space** means that S is an ordered pair (**X,T**) where **X** is a set and **T** is a topology for **X**. If (**X,T**) is a topological space S, then **U** is an **open set** in S if and only if **U** is an element of **T**, **p** is a **point** of S if and only if **p** is an element of **X,** and **A** is a **point set** of S if and only if **A** is a subset of **X**.

Example 1: Let R be the set of all real numbers. An **open number set U**

is a subset of R such that if **x** is an element of **U** then there is an open interval which contains **x** and is contained in **U**. Let **T** be the collection of all open number sets of R Then (R **T**) is a topological space and **T** is called the **usual topology** for R This topological space is denoted by **E1.**

Example 2: Let **X** be a set and let **T** be the collection of all subsets of **X**. Then (**X,T**) is a topological space. The collection **T** is called the **discrete topology** for **X**.

Example 3: Let **X** be a set and let **T** be **{ X**,**Ø.}** Then (**X,T**) is a topological space. The collection **T** is called the i**ndiscrete topology** for **X**.

Example 4: Let **X** be a set. **A** topology **T** is defined for **X** as follows: Suppose **U** is a subset of X; then **U** is an element of **T** if and only if either (i) **U** = **X** or (ii) **U** = **Ø.** or (iii) there exists a finite set **F** in **X** such that **U** = **X - F**. Then (**X,T**) is a topological space. The collection **T** is called the **finite complement topology** for **X**.

Example 5: Let Rn be the set of all *n*-tuples of real numbers. Recall that if **x** = $(x_1,x_2,\ldots,x_n)$ and **y** = $(y_1,y_2,\ldots,y_n)$ then the distance **d(x,y)** between **x** and **y** is usually defined as

$$\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+..+(x_n-y_n)^2}$$

$$(y_1,y_2,\ldots,y_n)$$

d(x,y) =

Then a topology **T** for Rn is defined as follows: A subset **U** of Rn belongs to **T** if and only if for each point **p** of **U**, there is a positive number **r**, such that **{x**: **d**(**p**,**x**) < **r}** is a subset of **U**. Then (Rn, **T**) is a topological space. The collection **T** is called the **usual topology for** Rn and this topological space is denoted by **En.**

Example 6: Let **X** = **{** a,b,c **}** be a set with three distinct elements. Let **T** = **{ Ø, X, {** a**}**, **{** b,c**} }** . Then (**X,T**) is a topological space.

Example 7: Let **X** = R. A topology **T** is defined for **X** as follows: A subset **U** of **X** belongs to **T** if and only if **U** = **Ø** or **U** contains an interval of the form [*a*,*b*) where [*a*,*b*) = **{x**: **x** is in R and $a \le$ **x** < *b***}** . The space (R,**T**) is called **E1 bad**.

Example 8: Let **X** be the subset of the plane consisting of all points **x** = (*a*, *b*) such that $b \ge 0$ and let **L** be the subset of **X** consisting of all points **x** = (*a*,0).

If **p** is an element of **X** - **L**, then a "neighborhood of **p**" is defined to be the interior of a circle lying entirely in **X** - **L** and centered at **p**.

If **q** is an element of **L**, then a "neighborhood of **q** " is defined to be the set consisting of **q** together with the interior of the upper half of a circle lying entirely in **X** - **L** and centered at **q** .

A topology **T** for **X** can be defined as follows: a subset **U** of **X** belongs to **T** if and only if whenever **x** is an element of **U**, there is a neighborhood of **x** contained in **U**.

Example 9: Let **X** be the same point set as in Example 8. If **p** is an element of **X** - **L** then a "neighborhood of **p**" is the interior of a circle lying entirely in **X** - **L** and centered at **p**. If **q** is an element of **L**, then a "neighborhood of **q** " is the set consisting of **q** together with the interior of a circle tangent to **L** at **q** .

A topology **T** for **X** is defined as follows: a subset **U** of **X** belongs to **T** if and only if whenever **x** is an element of **U**, there is a neighborhood of **x** contained in **U**. (More examples appear after Theorem 49)

Definitions: Suppose that (**X,T**) is a topological space, **A** is a subset of **X** and **p** is an element of **X**. The statement that **p** is a **limit point** of **A** means that if **U** is an open set and **p** is an element of **U**, then **U** contains a point of **A** distinct from **p**. The set **A** is **closed** if and only if every limit point of **A** belongs to **A.** The **closure** of **A**, denoted **A*** or **Cl(A)**, is the set to which **p** belongs if and only if either **p** is an element of **A** or **p** is a limit point of **A.**

Theorem 11: If (**X,T**) is a topological space and **M** is a subset of **X,** then

(i) **M** is closed if and only if **X** - **M** is open, and
(ii) **M** is open if and only if **X** - **M** is closed.

Theorem 12: If (**X,T**) is a topological space, then the union of two closed sets is closed.

Theorem 13: If (**X,T**) is a topological space, and **B** is a collection of closed subsets of **X,** then ∩**{** z: z is an element of **B}** is a closed subset of **X**.

Problem 1: Find a topological space (**X,T**) with some collection **S** of open sets whose intersection is not open.

Theorem 14: If **M** is a point set in topological space, then **M**** = **M*** and therefore **M*** is closed.

Theorem 15: If **M** is a point set in a topological space, **K** is a subset of **M** and **p** is a limit point of **K**, then **p** is a limit point of **M**.

Theorem 16: Suppose that (**X,T**) is a topological space, **A** and **B** are subsets of **X, p** is an element of **X,** and **p** is a limit point of the set **A∕ B.** Then either **p** is a limit point of **A** or **p** is a limit point of **B.**

Theorem 17: Suppose that (**X,T**) is a topological space and that **A** and **B** are subsets of **X**. Then **(A ∪ B)* = A* ∪ B*.**

Theorem 18: Suppose that (**X,T**) is a topological space, **Y** is a subset of **X** and S = **{ U**: for some set **V** of **T**, **U** = **Y** ∩**V}** . Then S is a topology for **Y**.

Question 3: Suppose (**X,T**) is a topological space and **{ Ai}** is a collection of subsets of **X**. Does the closure of the union of the Ai's equal the union of the closures; that is, is **[∪ Ai]* = ∪ [Ai]*?**

Theorem 18 gives us a new method of constructing topological spaces. If (**X,**

T) is a topological space, and **Y** is a subset of **X,** then the **relative topology** for **Y** is **{ U**: for some **V** of **T**, **U** = **Y** $\cap$**V}** If (**X,T**) is a topological space then (**Y**,**S**) is a **subspace** of (**X,T**) if and only if **Y** is a subset of **X** and **S** is the relative topology for **Y**.

Definition. Suppose that (**X,T**) is a topological space. The statement that (**X, T**) is a **Hausdorff space** means that if **x** and **y** are distinct points of **X,** then there exist disjoint open sets in **T**, one of which contains **x** and the other of which contains **y** .

Theorem 19: If **A** is a point set in a Hausdorff space, and **p** is a limit point of A, then every open set containing **p** contains an infinite subset of **A.**

Theorem 20: Every finite set in a Hausdorff space is closed.

Definition. Suppose that (**X,T**) is a topological space. The statement that (**X, T**) is a **regular space** means that if **x** is a point of **X, C**, is a closed subset of (**X,T**) and **x** does not belong to **C**, then there exist open sets **U** and **V** in (**X,T**) such that **x** belongs to **U**, **C** is a subset of **V**, and **U** and **V** are disjoint.

Question 4: Is there a Hausdorff space which is not a regular space?

Definitions: The statement that **f** is a **sequence** means that **f** is a function whose domain is **J** . Subscript notation may be used to denote the functional values of a sequence; thus **f**n is **f**(n). If **f** is a sequence and n is a positive integer, then the **nth term** of the sequence **f** is **f**(n).

Suppose that (**X,T**) is a topological space and **f** is a sequence such that (range **f**) is a subset of **X**. Then the statement that **f converges** to **p** means that **p** is an element of **X,** and if **U** is an open set containing **p**, then there exists a positive integer **N** such that for any integer m greater than **N,** it is the case that **f**(m) is in **U**.

Theorem 21: If (**X,T**) is a Hausdorff space, **f** is a sequence such that (range **f**) is a subset of **X, f** converges to **p**, and **f** converges to **q** , then **p** = **q** .

Question 5: Is Theorem 21 true if (**X,T**) is not Hausdorff?

Theorem 22: Suppose that **f** is a sequence such that (range **f**) is an infinite subset of a Hausdorff space **X,** and **f** converges to **p**. Then **p** is the only limit point, in **X,** of the point set (range **f**).

Question 6: Suppose that **X** is a Hausdorff space, **A** is point set in **X,** and **p** is a limit point of **A.** Does there exist a one-to-one sequence **f** such that (range **f**) is a subset of **A** and **f** converges to **p**?

Definitions. Suppose that **C** is a point set in a topological space and **G** is a collection, each element of which is a point set in the space. Then the statement that **G covers C**, or that **G** is a **covering** of **C** means that if **p** is an element of **C**, then there is an element **U** of **G** such that **p** is an element of **U**. The statement that **G** is an **open covering of C** means that (i) **G** covers **C**, and (ii) each element of **G** is an open set.

Suppose that **X** is a topological space, and **A** is a point set in **X**. The statement that **A** is **weakly countably compact** means that every infinite subset of **A** has a limit point in **A.** The statement that **A** is **compact** means that if **G** is any open covering of **A** then there is a subcollection **G'** such that (i) **G'** is finite, and (ii) **G'** covers **A.**

Theorem 23: If **X** is a topological space, and **A** is a compact point set in **X,** then **A** is weakly countably compact.

Question 7: If **X** is a topological space and **A** is a weakly countably compact point set in **X,** is **A** compact?

Theorem 24: If **A** is a compact point set in a Hausdorff space, then **A** is closed.

Theorem 25: Suppose that **X** is a topological space, **A** is a compact point set in **X** and **C** is a closed point set in **X** such that **C** is a subset of **A.** Then **C** is compact.

Theorem 26: Let **X** be the subset of **E1** consisting of 0,1,1/2, 1/3,...,l/ *n*,... . Then **X** is compact.

Theorem 27: In **E1,** let **I** be [0,1]. Then **I** is weakly countably compact.

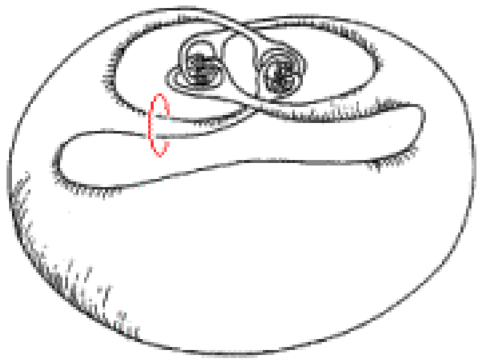
Theorem 28: In **E1**, let **I** be [0,1]. Then **I** is compact.

Definition. Suppose that (**X,T**) is a topological space and **B** is a subset of **T**. We call **B** a **basis** for the topology **T** if whenever **U** is an element of **T** and **p** is an element of **U**, then there is a set **V** in **B** such that **p** is an element of **V** and **V** is a subset of **U**.

Theorem 29: Suppose that **X** is a set and **B** is a collection of subsets of **X** whose union is **X**. Then **B** is a basis for some topology for **X** if and only if whenever **U** and **V** are sets in **B** and **p** is an element in **U** ∩**V**, then there exists a set **W** in **B** such that **p** is an element of **W** and **W** is a subset of **U** ∩**V**.

Definitions. If **X** is a topological space and each of **A** and **B** is a point set in **X,** then **A** and **B** are **separated** if and only if **A** and **B** are disjoint and neither contains a limit point of the other. Equivalently, **A** and **B** are separated if and only if (1) **A* ∩B = Ø** and (2) **A ∩B* = Ø.** If **X** is a topological space, and **M** is a point set in **X,** then **M** is **connected** if and only if **M** is not the union of two non-empty separated point sets. Equivalently, if we regard **M** as a subspace of **X,** then **M** is connected if and only if the only sets which are both open and closed in **M** are **M** and **Ø.**

Theorem 30: **E1** is connected.



Problem 2: In Problem 2 all point sets are in **E2** .
Show that:
(*a*) Every straight line is connected.
(*b*) Every open circular disk (the interior of a disk) is connected.
(*c*) Every closed circular disk (the union of a circle and its interior) is connected.
(*d*) If **C** is a circle, **x** and **y** are distinct points of **C**, then **C** and **C** - **{ x}** are connected, but **C** - **{** x,y**}** is not connected.
(*e*) If K is a circle, I is its interior and L is a subset of K, then I ⁄ L is connected
(*f*) Let **S** be **{** (x,y): $0 < x \le 1$, $y = \sin 1/x$**}** ⁄ **{**$(0,y)$:$-1 \le y \le 1$**}** . Then **S** is connected.

Theorem 31: If **X** is a topological space, **A** and **B** are separated point sets in **X,** and **F** is a connected subset of **A** ∪ **B,** then either **F** is a subset of **A** or **F** is a subset of **B.**

Theorem 32. If **M** is a connected point set in a topological space, and **N** is a set each point of which is a limit point of **M**, then **M** ∪ **N** is connected. In particular, if **M** is connected, then **M*** is connected.

Theorem 33: If **M** is a connected point set in a Hausdorff space, and **M** contains two distinct points, then every point of **M** is a limit point of **M**.

Theorem 34: If **X** is a countable, regular Hausdorff space containing more than one point, then **X** is not connected.

Theorem 35: If, in a topological space. **G** is a collection of connected point sets, and **p** is a point common to every set of G, then
$\cup$ **{ g: g** is an element of G **}** is connected.

Theorem 36: If, in a topological space, **G** is a collection of connected point sets, and there is a set **g** of G, such that each set of **G** intersects **g**, then $\cup$**{ h**: **h** is an element of G**}** is connected.

Theorem 37: If, in a topological space, **K** is a sequence of point sets such that for each positive integer *n*, *Kn* is connected and intersects *Kn+1*, then $\cup$**{** *Kn*: *n* belongs to **J }** is connected.

Theorem 38: If, in a topological space, **H** is a closed point set that is not connected, then **H** is the union of two disjoint non-empty closed point sets.

Problem 3: Give an example of a topological space (**X,T**) and a sequence **M1, M2, M3, ...** of closed connected point sets in **X** such that

(i) **Mn+1** is a subset of **Mn** for each n,
(ii) the intersection of all the **Mi** is non-empty, and
(iii) the intersection is not connected.

Definition: Suppose that (**X,T**) and (Y,S) are topological spaces and that **f** is a function whose domain is **X** and whose range is **Y**(that is, **f** is a function from **X** onto **Y**). The statement that **f** is **continuous** means that if **U** is an element of S, then **f**-1(**U**) is an element of T. Here **f**-1(**U**) denotes **{x**: **x** is in **X** and **f**(**x**) is in **U}** .

Theorem 39: Suppose that **X** and **Y** are topological spaces and **X** has the discrete topology. Then any function from **X** onto **Y** is continuous.

Theorem 40: Suppose that **X** and **Y** are topological spaces and that **f** is a function from **X** onto **Y**. Then the following statements are equivalent:

(1) **f** is continuous;

(2) if **C** is a closed set in **Y**. then **f**-1(**C**) is a closed set in X;

(3) if **x** is an element of **X** and **U** is an open set in **Y** containing **f**(**x**), then there exists an open set **V** in **X** such that **x** is an element of **V** and **f**(**V**) is a subset of **U**. .

Definitions: Suppose that **X** and **Y** are topological spaces and that **f** is a function from **X** onto **Y**. The statement that **f** is a **homeomorphism** means that

(i) **f** is one-to-one,

(ii) **f** is continuous, and

(iii) **f**-1 is continuous.

Definition: If **X** and **Y** are topological spaces, then the statement that **X** and **Y** are **homeomorphic spaces** means that there is a homeomorphism of **X** onto **Y**.

Theorem 41: Suppose that **X** is a connected topological space, **Y** is a topological space, and **f** is a continuous function from **X** onto **Y**. Then **Y** is connected.

Definitions: Suppose that **X** and **Y** are topological spaces and that **f** is a function from **X** onto **Y**. The statement that **f** is a **connected function** means that whenever **A** is a connected subset of **X,** then **f(A)** is a connected subset of **Y**. The statement that **f** is a c**ompact function** means that whenever **A** is a compact subset of **X,** then **f(A)** is a compact subset of **Y**.

Question 8: Suppose **X** and **Y** are topological spaces and that **f** is a function from **X** onto **Y**. If **f** is a connected function, is **f** necessarily continuous?

Theorem 42: If **X** is a weakly countably compact, Hausdorff topological space, **Y** is a topological space and **f** is a continuous function from **X** onto **Y**, then **Y** is weakly countably compact.

Theorem 43: Suppose that **X** is a compact topological space, **Y** is a topological space, and **f** is a continuous function from **X** onto **Y**. Then **Y** is compact.

Question 9: Suppose **X** and **Y** are topological spaces and that **f** is a function from **X** onto **Y**. If **f** is compact, is **f** necessarily continuous?

Theorem 44: Suppose that **X** is a compact space, **Y** is a Hausdorff space, and **f** is a one-to-one continuous function from **X** onto **Y**. Then **f** is a homeomorphism.

Definition: Suppose that **X** is a topological space. The statement that **X** is **normal** means that if **C** and **D** are disjoint closed sets in **X,** then there exist disjoint open sets **U** and **V** such that **C** is a subset of **U** and **D** is a subset of **V**.

Theorem 45: Suppose that **X** is a normal space and **C** and **D** are disjoint closed sets in **X**. Then there exist open sets **U** and **V** such that **C** is a subset of **U**, **D** is a subset of **V**, and **U*** ∩**V* = Ø.**

Theorem 46: Suppose that **X** is a compact Hausdorff space. Then **X** is regular.

Theorem 47: Suppose that **X** is a compact Hausdorff space. Then **X** is normal.

Definition: The **Cantor middle thirds set** is the subspace of **E1** described as follows:

Let M1 = [0,1],
M2 = [0,1/3] ∕ [2/3, 1],
M3 = [0,1/9]∕[2/9, 1/3]∕[2/3, 7/9]∕[8/9, 1],

and Mn is obtained from Mn-1 by removing the open middle third from each of the closed intervals in Mn-1. Then the Cantor middle thirds set **C** is the intersection of all the Mi, i = 1,2,3,... If **X** is a topological space, then the statement that **X** is a **Cantor set** means that **X** is homeomorphic with **C**.

Theorem 48: Suppose that **X** is a Cantor set. Then
(i) **X** is compact;
(ii) each point of **X** is a limit point of **X,** and
(iii) **X** is uncountable.

Theorem 49: Let **I** denote the subspace [0,1] of **E1**. Then there is a continuous function from **C** onto **I**.

Definition: If **X** is a set and **A** is a subset of **X,** then the **complement** of **A** denoted **X - A,** is the set of all elements of **X** which are not in **A.**

Example 10: Let **X** be the set of real numbers. If **p** is nonzero, then a "neighborhood of **p**" is defined to be any set containing **p**. If **p** = 0, then a "neighborhood of **p**" is a subset **V** of **X** such that 0 is an element of **V** and the set **X - V** is countable.
A topology for **X** is defined as follows: A subset **U** of **X** belongs to **T** if and only if whenever **x** is an element of **U**, there is a neighborhood of **x** contained in **U**.

Example 11. If **X** is any infinite set and **p** is a particular point of **X,** we can define a topology **T** for **X** as follows: **V** is an element of **T** if and only if **p** is an element of **X** - **V** or **X** - **V** is finite. (**X,T**) is called a **Fort space**.

Example 12: If **X** is any uncountable set and **p** is a particular point of **X,** we can define a topology **T** for **X** as follows: **V** is an element of **T** if and only if **p** is an element of **X** - **V** or **X** - **V** is countable. (**X,T**) is called a **Fortissimo space**.

Definition: Let (**X,T**) be a topological space. A subset **C** of **X** is a **component** of **X** if and only if **C** is a nonempty connected set with the property that if **D** is any connected subset of **X** such that **D** $\cap$**C** is nonempty, then **D** is a subset of **C**. A subset **G** of **X** is a **region** if and only if **G** is both connected and open in **X**.

Theorem 50: Let **M** be a subset of a topological space (**X,T**) and **p** a point of **M**. Then the component of **M** containing **p** is the union of all connected subsets of **M** that contain the point **p**.

Theorem 51: Let **M** be a subset of a topological space (**X,T**). Then every component of **M** is closed in **M**.

Theorem 52: Let **A** and **B** be components of a subset **M** of a topological space (**X,T**). If **A** and **B** are distinct, then **A** and **B** are disjoint.

Definitions: An **arc** is a topological space homeomorphic to the closed interval [0,1] in **E1.** By the **plane**, we will mean the topological space **E2** (See Example 5). A **simple closed curve** is a topological space homeomorphic to a circle **C** in the plane. The union **P** of a finite set of closed line segments in the plane is called a **polygonal set**. The end points of the line segments are the **vertices** of **P**. If **P** is also an arc, then **P** is a **polygonal arc**; if **P** is also a simple closed curve, then **P** is called a **polygon**. A subset **S** of the plane is **polygonally connected** if and only if each pair of points in **S** can be joined by a polygonal arc contained in **S**.

Theorem 53: A polygonal arc in the plane is connected.

Theorem 54: If **S** is a subset of the plane which is polygonally connected, then **S** is connected.

Theorem 55: An open subset of the plane is polygonally connected if and only if it is connected.

Definition: Let (**X,T**) be a topological space and let **A** be a subset of **X**. Then **A separates X** if and only if **X - A** is not connected.

Theorem 56: No point separates the plane.

Theorem 57: Every line separates the plane.

Theorem 58: The complement of a line in the plane has exactly two components.

Theorem 59: Every angle separates the plane and the complement of the angle has exactly two components.

Theorem 60: Every line, not passing through a vertex of a polygon **P** in the plane, intersects **P** in an even number of points.

Theorem 61: If **A** is an angle whose vertex is not on the polygon **P** in the plane, and whose rays do not pass through a vertex of **P**, then **A** intersects **P** in an even number of points.

Theorem 62: If **x** is a point not on the polygon **P** in the plane and if some ray from **x** which does not pass through a vertex of **P** intersects **P** in an odd number of points, then so also does every other ray from **x** which contains no vertices of **P**.

Theorem 63: Every polygon **P** separates the plane.

Theorem 64: No polygonal arc separates the plane.

Definition: If **U** is an open set, then the set **U* - U** is called the **boundary** of **U**.

Theorem 65: If **P** is a polygon in the plane, then **P** is the boundary of every component of the complement of **P**.

Theorem 66: The complement of a polygon in the plane has exactly two components.

Definitions: An **even chain** is a closed set **C** which is the union of a finite number of nonintersecting open segments, called **edges**, and points, called **vertices**, such that each vertex is on an even number of edges and such that each end point of an edge of **C** is a vertex of **C**.

Theorem 67: If the vertices **x** and **y** of an even chain **C** are joined in **C** by a polygonal arc **L** then if we omit from **C** all open edges lying in **L** , **x** and **y** can still be joined by a polygonal arc in the rest of **C**.

Theorem 68: No arc separates the plane.

Definition: **A semipolygon** is a simple closed curve which contains a line segment as a subset.

Theorem 69: Every semipolygon separates the plane into exactly two regions.

Theorem 70 (***Jordan Curve Theorem*****):** Every simple closed curve **J** separates the plane into exactly two regions and is the boundary of each of these regions.