The Minimum Size of Saturated Graphs

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Joint work with:

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- Tomasz Łuczak, Adam Mickiewicz University and Emory University

  Ongoing work with:
- Oleg Pikhurko, Carnegie Mellon University

Definition

A graph G is **F-saturated** if:

$F \not\subset G$

$F \subset G + e$ for any $e \in E(\overline{G})$

Problem

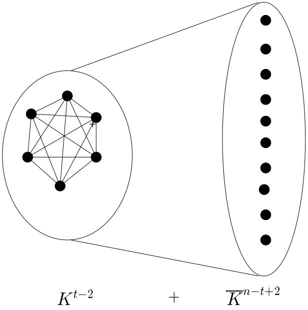
Determine the minimum number of edges, $sat(n, F)$*, of an* F*-saturated graph.*

History

Theorem (Erdős, Hajnal, Moon - 1963)

$$sat(n, K^t) = (t-2)(n-1) - \binom{t-2}{2}$$

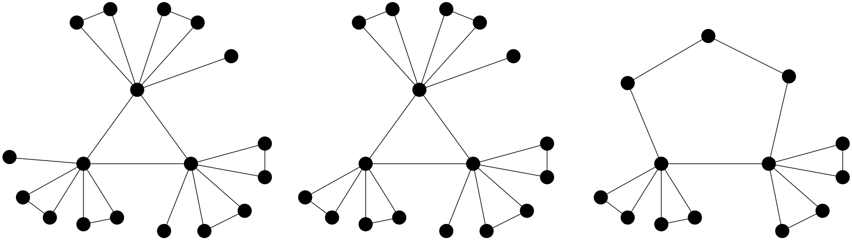
Furthermore, the only K^t-saturated graph with this many edges is $K^{t-2} + \overline{K}^{n-t+2}$.



History

Theorem[Ollmann - '72, Tuza - '86]

$$sat(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor, \quad n \geq 5$$



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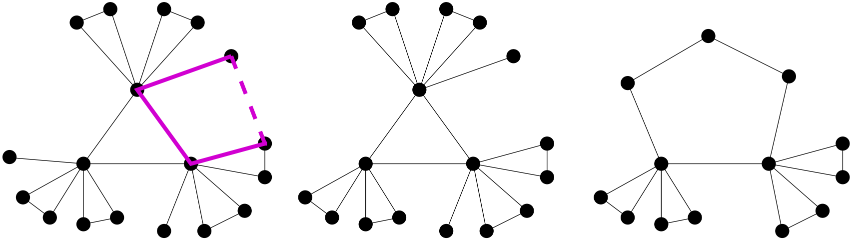

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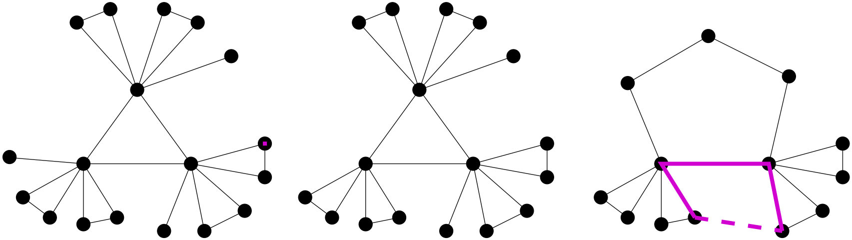


Other Subgraphs

Exact values of $sat(n, F)$ known for:

- **matchings** (Mader - '73),
- **paths and stars** (Kászonyi and Tuza - '86),
- **hamiltonian cycle,** C_n (Clark et al. '86-'92)

$$sat(n, C_n) = \lfloor \frac{3n+1}{2} \rfloor, n \geq 53.$$

Other Subgraphs

Recent progress made on:

- **five cycle,** C_5 (Y.C. Chen '05+)

$$sat(n, C_5) = \lceil \frac{10n - 10}{7} \rceil, n \geq 21$$

- **hamiltonian path,** P_n (Frick and Singleton, 05)

$$sat(n, P_n) = \lceil \frac{3n - 2}{2} \rceil, n \geq 54$$

- **longest path = detour**(Beineke, Dunbar, Frick, '05)

Difficulties and Hereditary Properties Lacking

Quote from Erdős, Hajnal and Moon:

"One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult."

- $sat(n, F) \not\leq sat(n+1, F)$
- $\mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow sat(n, \mathcal{F}_1) \geq sat(n, \mathcal{F}_2)$
- $F' \subset F \not\Rightarrow sat(n, F') \leq sat(n, F)$

Best known upper bound

Theorem (Kászonyi L. and Tuza, Z.)

Let $\mathcal{F}$ be a family of non-empty graphs. Set

$$u = min\{|U| : F \in \mathcal{F}, U \subset V(F), F - U \text{ is a star}\},$$

$$s = min\{e(F - U) : F \in \mathcal{F}, U \subset V(F), F - U \text{ is a star and } |U| = u\}.$$

Furthermore, let p be the minimal number of vertices in a graph $F \in \mathcal{F}$ for which the minimum s is attained. If $n \geq p$ then

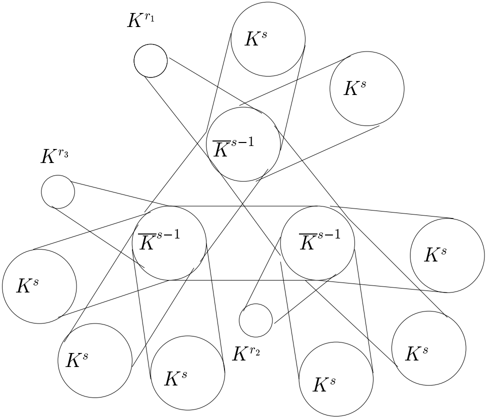
$$sat(n, \mathcal{F}) \leq (u + \frac{s-1}{2})n - \frac{u(s+u)}{2}.$$

O. Pikhurko generalized this result to r-uniform hypergraphs, $\mathcal{H}$, showing that $sat(n, \mathcal{H}) < O(n^{r-1})$.

Best Known Lower Bound

????

Saturation for Bipartite Graphs, $K_{s,s}$



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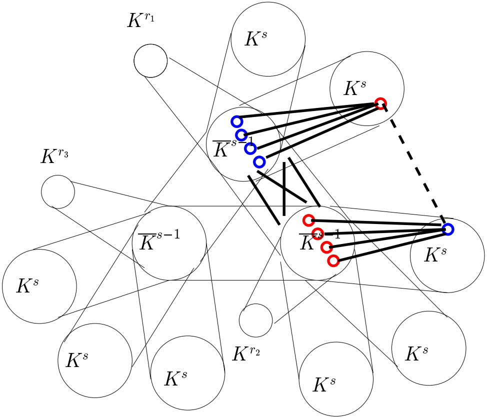

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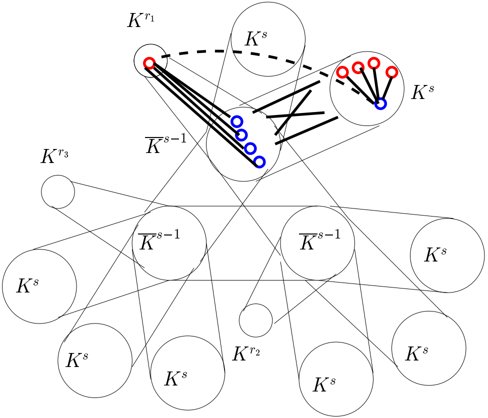

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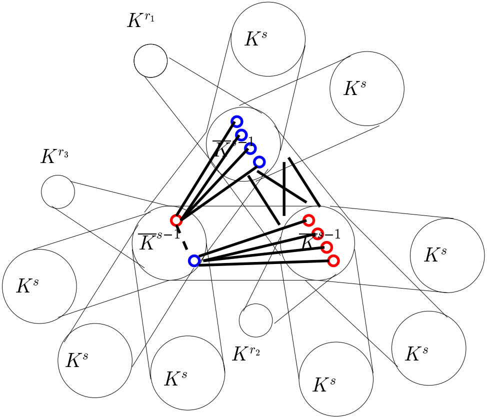


Theorem

For $n \geq 3s - 3$,

$$sat(n, K_{s,s}) \leq \lfloor \frac{(3s-3)n - (2s-1)(s-1)}{2} \rfloor - (s-1).$$

Theorem

For $n \geq st + s - 3$,

$$sat(n, K_s^t) \leq \lfloor \frac{(2st - s - 3)n - s^2t^2 + s^2t + 2st - s - 1}{2} \rfloor - (s-1).$$

(This can be improved a little in the $s = 2$ case.)

Saturation for Bipartite Graphs

Ongoing with O. Pikhurko:

$$2n - O(\sqrt{n}) \leq sat(n, K_{3,3}) \leq \lfloor 3n - 7 \rfloor$$

Conjecture

The construction given is asymptotically optimal.

Saturation for Cycles

- **Conjecture (Bollobás - '78)**

$$n + c_1 \frac{n}{l} \leq sat(n, C_l) \leq n + c_2 \frac{n}{l}$$

- Theorem (Barefoot et.al - '96)

$$(1 + \frac{1}{2l+8})n \leq sat(n, C_l)$$

Saturation for Cycles

Theorem (Barefoot et.al. - '96)

$sat(n, C_l) \leq (1 + \frac{6}{l-3})n + O(l^2)$ *for* l *odd*, $l \geq 9$

$sat(n, C_l) \leq (1 + \frac{4}{l-2})n + O(l^3)$ *for* l *even*, $l \geq 14$

Saturation for Cycles

Theorem (Barefoot et.al. - '96)

[Gould, Łuczak, S.]

$sat(n, C_l) \leq (1 + \frac{1}{3}\frac{6}{l-3})n + \frac{5l^2}{4}$ *for l odd,* $l \geq 9$, $l \geq 17, n \geq 7l$

$sat(n, C_l) \leq (1 + \frac{1}{2}\frac{4}{l-2})n + \frac{5l^2}{4}$ *for l even* $l \geq 14$, $l \geq 10, n \geq 3l$

Theorem

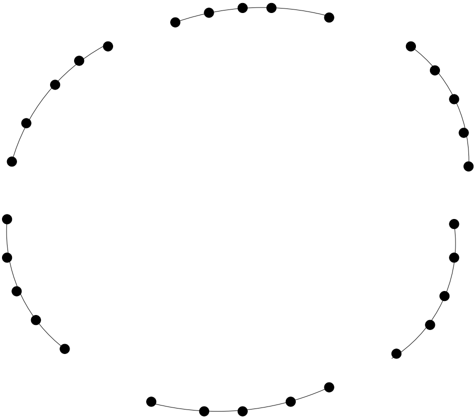
[Gould, Łuczak, S.] For $l = 8, 9, 11, 13$ *or* 15 *and* $n \geq 2l$

$$\begin{aligned} sat(n, C_l) &\leq \left\lceil \frac{3n + l^2 - 9l + 15}{2} \right\rceil \\ &< \left\lceil \frac{3n}{2} \right\rceil + \frac{l^2}{2} \end{aligned}$$

Our Inspiration

Lance Armstrong's Trek Madone SSL proto, 12/06/2004. The complete special edition Bontrager front wheel with super-minimal 19mm tubulars.

Logic of First Construction



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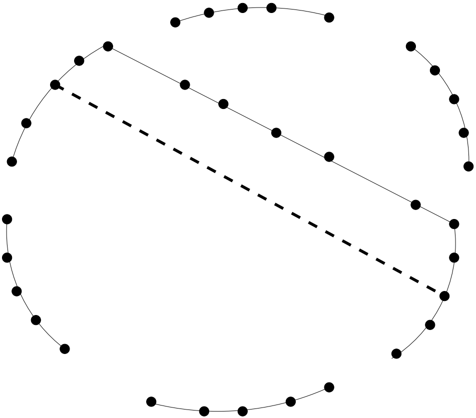

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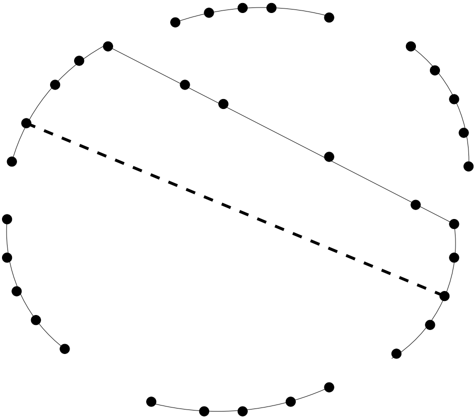
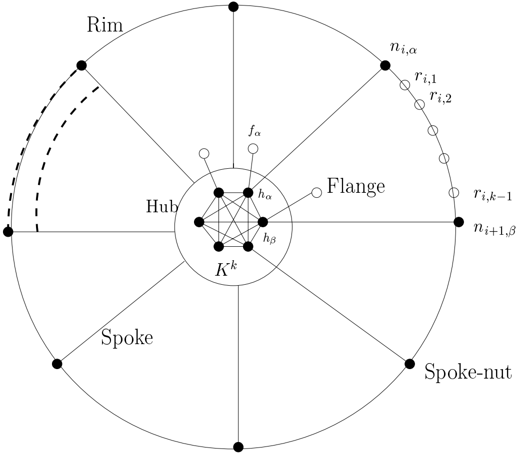


The Even Łuczak Wheel, $l = 2k + 2 \geq 10$



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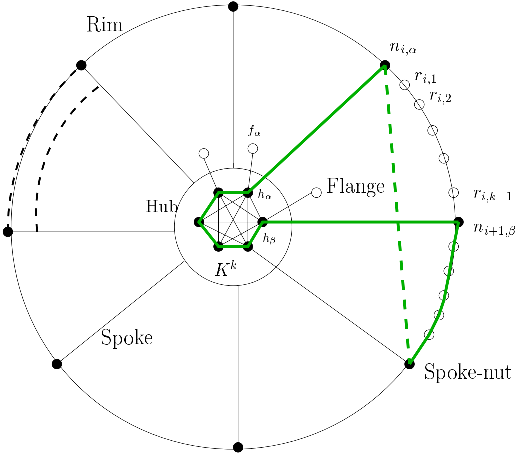

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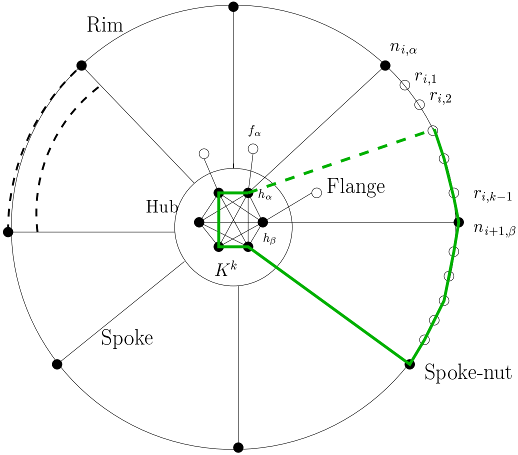

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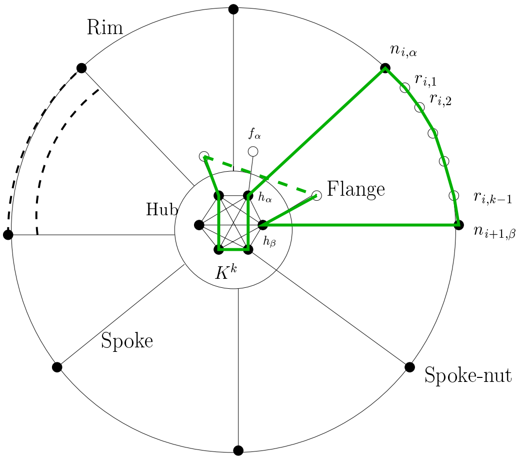


Counting Edges of the Łuczak Wheel

For $l = 2k + 2$

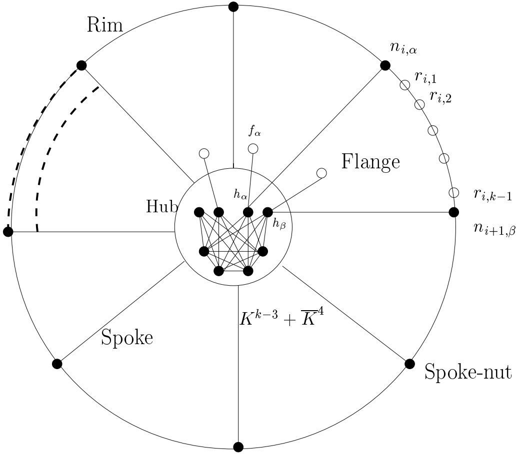
$$|E(L-Wheel)| \quad = \quad \overbrace{(n-k-a)}^{Rim} + \overbrace{\frac{n-k-a}{k}}^{Spokes} + \overbrace{a + \Sigma_{i=1}^{k}\binom{a_i}{2}}^{Flange} + \overbrace{\binom{k}{2}}^{Hub}.$$

Theorem

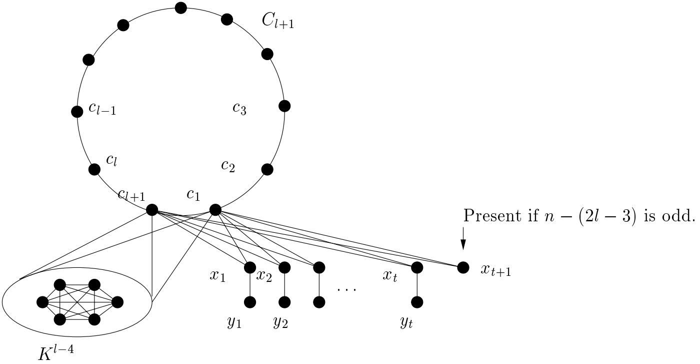
For $k \geq 4,\ l = 2k+2,\ n \equiv a \mod k$ and $n \geq 3l$,

$$\begin{aligned} sat(n, C_l) &\leq n(1+\frac{1}{k}) + \frac{k^2-3k-2}{2} - \frac{a}{k} + \Sigma_{i=1}^{k}\binom{a_i}{2} \\ &\leq n(1+\frac{2}{l-2}) + \frac{5l^2}{4}. \end{aligned}$$

The Odd Łuczak Wheel, $l = 2k + 3 \geq 17$



Another Construction, $l \geq 5$



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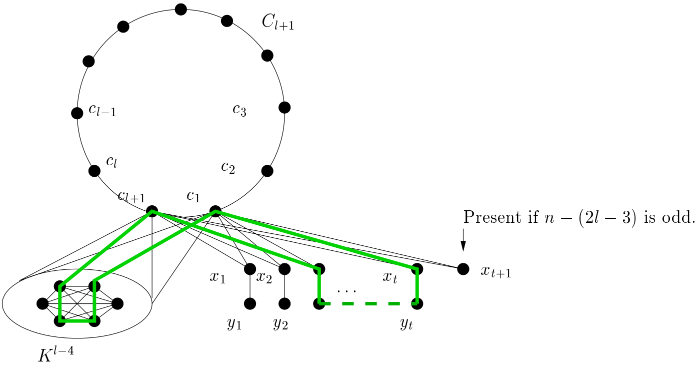

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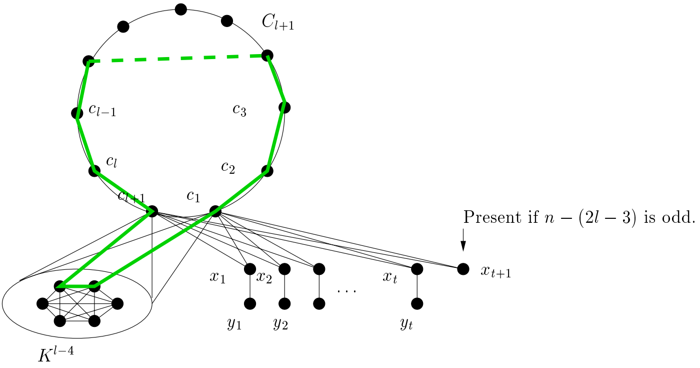

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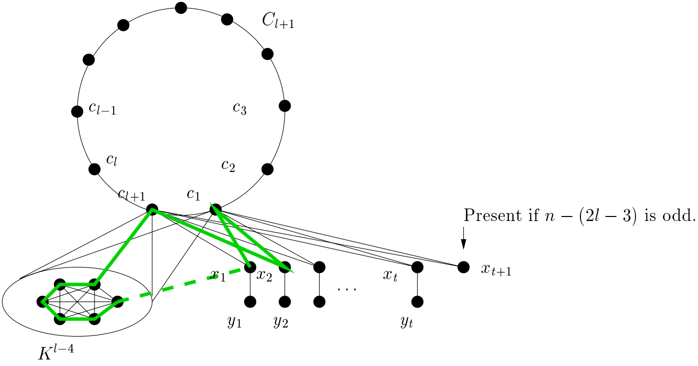

Summary of Results on Cycles

C_l-saturated graphs of minimum size			
l	$sat(n, C_l)$	$n \geq$	Reference
3	$= n - 1$	3	[8]
4	$\lfloor \frac{3n-5}{2} \rfloor$	5	[10, 11]
5	$\lceil \frac{10n-10}{7} \rceil$	21	[4]
6	$\leq \frac{3n}{2}$	11	[1]
7	$\leq \frac{7n+12}{5}$	10	[1]
8,9,11,13,15	$\leq \frac{3n}{2} + \frac{l^2}{2}$	$2l$	Theorem 10
≥ 10 and $\equiv 0 \mod 2$	$\leq \left(1 + \frac{2}{l-2}\right) n + \frac{5l^2}{4}$	$3l$	Theorem 11
≥ 17 and $\equiv 1 \mod 2$	$\leq \left(1 + \frac{2}{l-3}\right) n + \frac{5l^2}{4}$	$7l$	Theorem **??**
n	$\lfloor \frac{3n+1}{2} \rfloor$	20	[5, 6, 7, **?**]

Table: A Summary of Results for $sat(n, C_l)$

A few questions

- Problem

  Are any of these constructions optimal? Can one improve the lower bounds?

- Problem (Hanson, Toft)

  Is there a relation with sat$(n, \mathcal{F})$ *and the ramsey number of* $\mathcal{F}$.

A few questions

- Problem (Barefoot et.al. - '96)

  Determine the value of l which minimizes $sat(n, C_l)$ for fixed n.

- Problem (Pikhurko)

  See paper "Results and Open Problems on Minimum Saturated Hypergraphs", Ars Combin.

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