Recent Results and Open Problems on the Minimum Size of Saturated Graphs

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joint work with
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Definition

A graph G is ***F*-saturated** if;

$F \not\subset G$

$F \subset G + e$ for any $e \in E(\overline{G})$

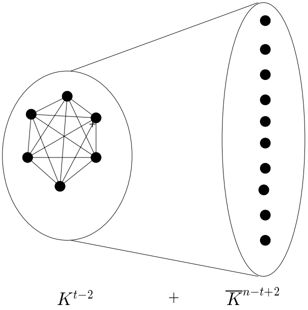
Problem

Determine the minimum number of edges, $sat(n, F)$, of an F-saturated graph.

Theorem (Erdős, Hajnal, Moon - 1964)

$$sat(n, K^t) = (t-2)(n-1) - \binom{t-2}{2}$$

Furthermore, the only K^t-saturated graph with this many edges is $K^{t-2} + \overline{K}^{n-t+2}$.



Let $sat(n, F, \delta)$ equal *minimum* number of edges in a graph on n vertices and minimum degree δ that is F-saturated.

Theorem (Duffus, Hanson - '86)

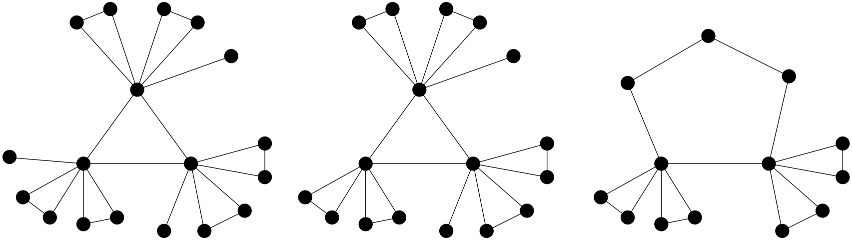
$$sat(n, K_3, 2) = 2n - 5, \quad n \geq 5$$
$$sat(n, K_3, 3) = 3n - 15, \quad n \geq 10$$

Problem (Bollobás - '95)

Is it true that for every fixed $\delta \geq 1$ *one has* $sat(n, K_3, \delta) = \delta n - O(1)$

Theorem (Ollmann - '72, Tuza - '86)

$$sat(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor, \quad n \geq 5$$



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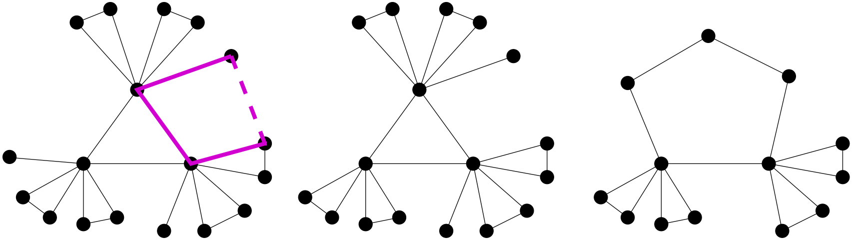

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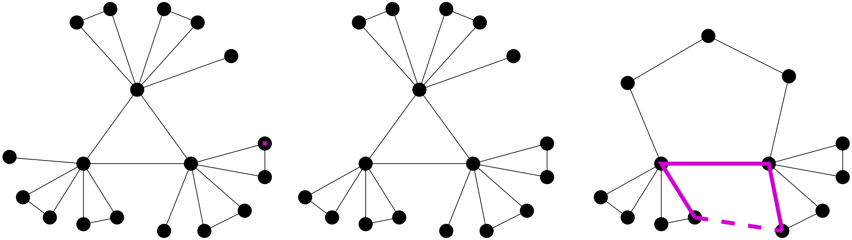


Theorem (Fisher, Fraughnaugh, Langley - '97)

$$sat(n, P_3 - connected) = \lfloor \frac{3n-5}{2} \rfloor$$

Theorem (Pikhurko, S.)

There exists a constant C such that for all $n \geq 5$,

$$2n - Cn^{3/4} \leq sat(n, K_{2,3}) \leq 2n - 3$$

Theorem (Fisher, Fraughnaugh, Langley, -'95)

$$sat(n, C_5) \leq \lceil \frac{10n - 10}{7} \rceil, n \neq 4$$

Theorem (Y.C.Chen)

$$sat(n, C_5) = \lceil \frac{10n - 10}{7} \rceil, n \geq 21$$

Problem (FFL)

Determine $sat(n, P_4 - connected)$.

Hamiltonian Cycles

Theorem

$$sat(n, C_n) = \lfloor \frac{3n+1}{2} \rfloor, n \geq 53$$

Bondy ('72) showed the lower bound. Clark, Entringer, Crane and Shapiro ('83-'86) gave upper bound based on Isaacs' flower snarks (girth 5, 6). L. Stacho ('96) gave further constructions based on the Coxeter graph (girth 7).

Problem (Horák, Širáň -'86)

Is there a maximally non-hamiltonian graph of girth at least 8?

- **Conjecture (Bollobás - '78)**

$$n + c_1\frac{n}{l} \leq sat(n, C_l) \leq n + c_2\frac{n}{l}$$

- Theorem (Barefoot, Clark, Entringer, Porter, Székely, Tuza - '96)

$$(1 + \frac{1}{2l+8})n \leq sat(n, C_l)$$

Theorem (Barefoot et al. - '96)

$$sat(n, C_l) \leq (1 + \frac{6}{l-3})n + O(l^2) \text{ for } l \text{ odd, } l \geq 9$$

$$sat(n, C_l) \leq (1 + \frac{4}{l-2})n + O(l^3) \text{ for } l \text{ even, } l \geq 14$$

Theorem (Barefoot et al. - '96)

[Gould, Łuczak, S. -'06]
$sat(n, C_l) \leq (1 + \frac{1}{3}\frac{6}{l-3})n + \frac{5l^2}{4}$ *for* l *odd,* $l \geq 9$, $l \geq 17, n \geq 7l$

$sat(n, C_l) \leq (1 + \frac{1}{2}\frac{4}{l-2})n + \frac{5l^2}{4}$ *for* l *even* $l \geq 14$, $l \geq 10, n \geq 3l$

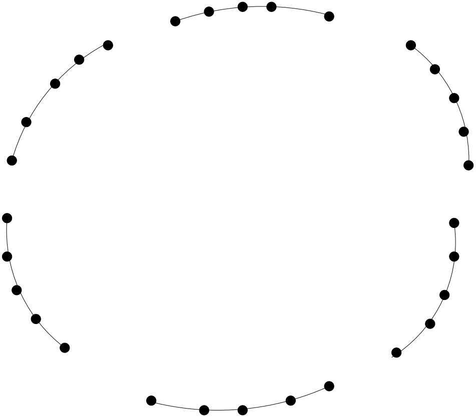
Theorem

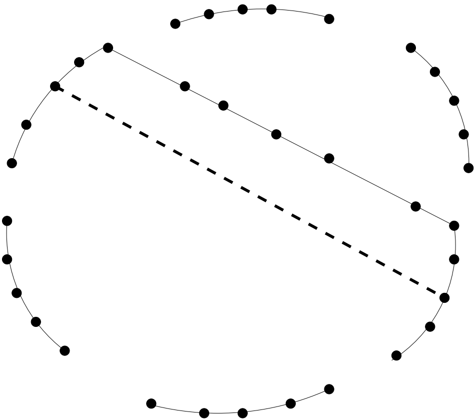
[Gould, Łuczak, S. -'06] For $l = 8, 9, 11, 13$ *or* 15 *and* $n \geq 2l$

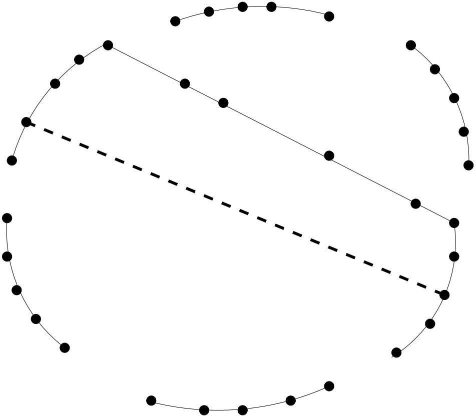
$$\begin{aligned} sat(n, C_l) &\leq \left\lceil \frac{3n + l^2 - 9l + 15}{2} \right\rceil \\ &< \left\lceil \frac{3n}{2} \right\rceil + \frac{l^2}{2} \end{aligned}$$

Our Inspiration

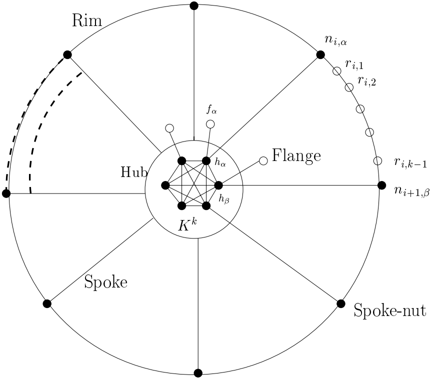
Lance Armstrong's Trek Madone SSL proto, 12/06/2004. The complete special edition Bontrager front wheel with super-minimal 19mm tubulars.







The Even Łuczak Wheel, $l = 2k + 2 \geq 10$



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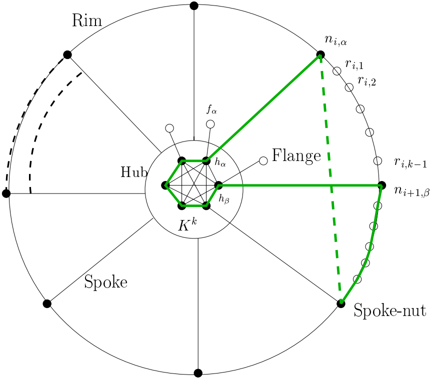

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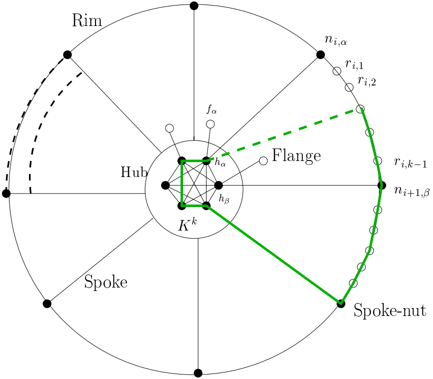

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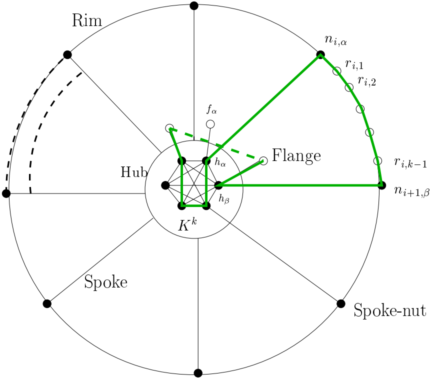


Counting Edges of the Łuczak Wheel

For $l = 2k + 2$

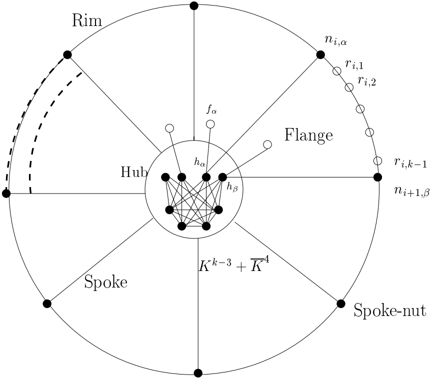
$$|E(L-Wheel)| = \overbrace{(n-k-a)}^{Rim} + \overbrace{\frac{n-k-a}{k}}^{Spokes} + \overbrace{a}^{Flange} + \overbrace{\binom{k}{2}}^{Hub}.$$

Theorem

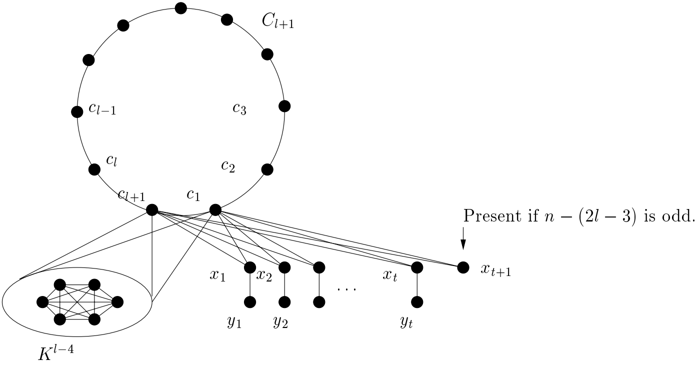
For $k \geq 4$, $l = 2k+2$, $n \equiv a \mod k$ and $n \geq 3l$,

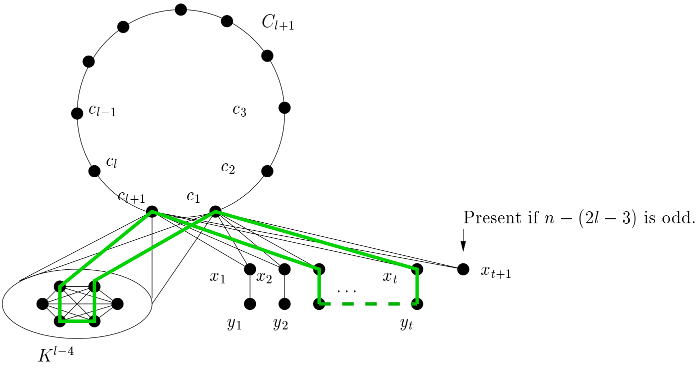
$$\begin{aligned} sat(n, C_l) &\leq n\left(1+\frac{1}{k}\right) + \frac{k^2-3k-2}{2} - \frac{a}{k} \\ &\leq n\left(1+\frac{2}{l-2}\right) + \frac{5l^2}{4}. \end{aligned}$$

The Odd Łuczak Wheel, $l = 2k + 3 \geq 17$



Inspired by Fisher, Fraughnaugh and Langley




C_{l+1}
c_{l-1}
c_3
c_l
c_2
c_{l+1}
c_1
Present if $n-(2l-3)$ is odd.
x_1
x_2
x_t
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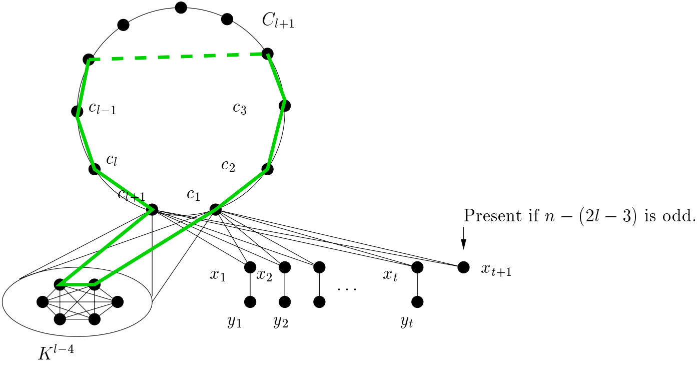
y_t
$\cdots$
K^{l-4}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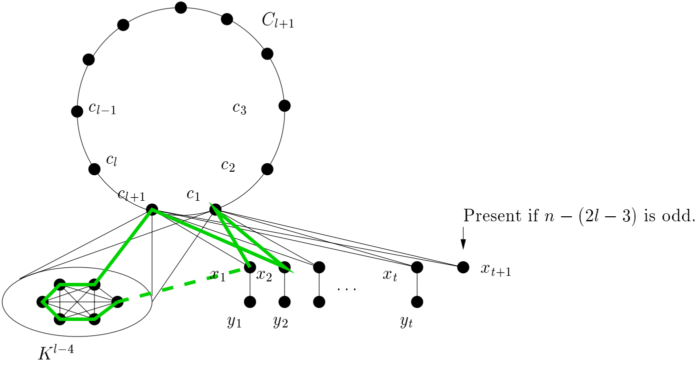
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C_l-saturated graphs of minimum size			
l	$sat(n, C_l)$	$n \geq$	Reference
3	$= n - 1$	3	EHM
4	$= \lfloor \frac{3n-5}{2} \rfloor$	5	Ollmann; Tuza
5	$= \lceil \frac{10n-10}{7} \rceil$	21	FFL; Chen
6	$\leq \frac{3n}{2}$	11	Barefoot et al.
7	$\leq \frac{7n+12}{5}$	10	Barefoot et al.
8,9,11,13,15	$\leq \frac{3n}{2} + \frac{l^2}{2}$	$2l$	GLS
≥ 10 and $\equiv 0 \mod 2$	$\leq \left(1 + \frac{2}{l-2}\right)n + \frac{5l^2}{4}$	$3l$	Łuczak wheel
≥ 17 and $\equiv 1 \mod 2$	$\leq \left(1 + \frac{2}{l-3}\right)n + \frac{5l^2}{4}$	$7l$	Łuczak wheel
n	$\lfloor \frac{3n+1}{2} \rfloor$	20	Bondy; CE, CES

Problem (Barefoot et al. - '96)

Determine the value of l which minimizes $sat(n, C_l)$ for fixed n.

Problem

Are any of these constructions optimal? Can one improve the lower bound?

Other Subgraphs

Other values of $sat(n, F)$ known for:

- **matchings** (Mader - '73),
- **paths and stars** (Kászonyi and Tuza - '86),
- **hamiltonian path, P_n** (Frick and Singleton, 05; Dudek, Katona, Wojda - '06)

$$sat(n, P_n) = \lceil \frac{3n-2}{2} \rceil, n \geq 54$$

- **longest path = detour**(Beineke, Dunbar, Frick, '05)

Difficulties and Hereditary Properties Lacking

Quote from Erdős, Hajnal and Moon:
"One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult."

- $sat(n, F) \not\leq sat(n+1, F)$
- $\mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow sat(n, \mathcal{F}_1) \geq sat(n, \mathcal{F}_2)$
- $F' \subset F \not\Rightarrow sat(n, F') \leq sat(n, F)$

Best known upper bound

Theorem (Kászonyi L. and Tuza, Z.)

Let F be a graph. Set

$$u = |V(F)| - \alpha(F) - 1$$

$$s = min\{e(F') : F' \subseteq F, \alpha(F') = \alpha(F), |V(F')| = \alpha(F) + 1\}.$$

Then

$$sat(n, F) \leq (u + \frac{s-1}{2})n - \frac{u(s+u)}{2}.$$

They considered a clique on u vertices joined to an $(s-1)$-regular graph.

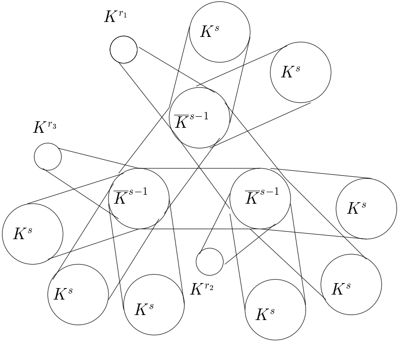
Best Known Lower Bound

????

Problem

For an arbitrary graph F, determine a non-trivial lower bound on $sat(n, F)$.

Saturation for Bipartite Graphs, $K_{s,s}$



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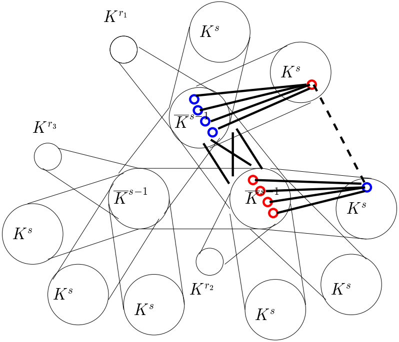

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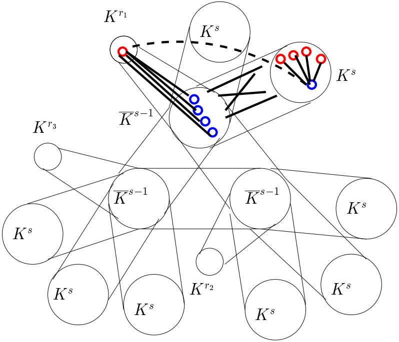

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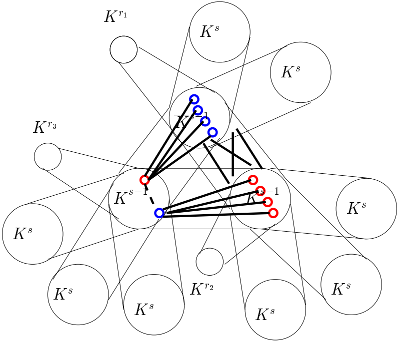

Theorem (S. - '05)

For $n \geq 3s-3$,

$$sat(n, K_{s,s}) \leq \lfloor \frac{(3s-3)n-(2s-1)(s-1)}{2} \rfloor - (s-1).$$

Theorem (S. - '05)

For $n \geq st+s-3$,

$$sat(n, K_s^t) \leq \lfloor \frac{(2st-s-3)n-s^2t^2+s^2t+2st-s-1}{2} \rfloor - (s-1).$$

Theorem (Gould, S. - 06+)

For integers $t \geq 3$, $n \geq 4t-4$,

$$sat(n, K_2^t) \leq sat(n, K_2^t, 2t-3) = \lceil \frac{(4t-5)n - 4t^2 + 6t - 1}{2} \rceil.$$

Problem

Given a fixed graph F, for n sufficiently large determine if the function $sat(n, F, \delta)$ is monotonically increasing in δ.

And Ramsey Numbers

$F \to (F_1, \ldots, F_t)$ if any t coloring of $E(F)$ contains a monochromatic F_i-subgraph of color i for some $i \in [t]$.

Conjecture (Hanson and Toft, '87)

Given $t \geq 2$ and numbers $m_i \geq 3, i \in [t]$, let

$$\mathcal{F} = \{F : F \to (K_{m_1}, \ldots, K_{m_t})\}.$$

Let $r = r(K_{m_1}, \ldots, K_{m_t})$ be the classical Ramsey number. Then

$$sat(n, \mathcal{F}) = (r-2)(n-1) - \binom{r-2}{2}.$$

Many Thanks!!

Problem (Pikhurko)

For even more problems see paper "Results and Open Problems on Minimum Saturated Hypergraphs", Ars Combin.

Talk and results are available online at:
http://community.middlebury.edu/∼jschmitt/