Recent Results and Open Problems on the Minimum Size of Saturated Graphs

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joint work with
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Definition
A graph $G$ is $F$-saturated if;
$F \not\subseteq G$

$F \subset G + e$ for any $e \in E(\overline{G})$

Problem
Determine the minimum number of edges, $\text{sat}(n, F)$, of an $F$-saturated graph.
Theorem (Erdős, Hajnal, Moon - 1964)

\[ \text{sat}(n, K^t) = (t - 2)(n - 1) - \binom{t - 2}{2} \]

Furthermore, the only \( K^t \)-saturated graph with this many edges is \( K^{t-2} + \overline{K}^{n-t+2} \).
Let $sat(n, F, \delta)$ equal *minimum* number of edges in a graph on $n$ vertices and minimum degree $\delta$ that is $F$-saturated.

**Theorem (Duffus, Hanson - '86)**

$$sat(n, K_3, 2) = 2n - 5, \quad n \geq 5$$

$$sat(n, K_3, 3) = 3n - 15, \quad n \geq 10$$

**Problem (Bollobás - '95)**

*Is it true that for every fixed $\delta \geq 1$ one has*

$$sat(n, K_3, \delta) = \delta n - O(1)$$
Theorem (Ollmann - '72, Tuza - '86)

\[
sat(n, C_4) = \left\lfloor \frac{3n - 5}{2} \right\rfloor, \quad n \geq 5
\]
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Theorem (Fisher, Fraughnaugh, Langley - '97)

\[ sat(n, P_3 - connected) = \left\lfloor \frac{3n - 5}{2} \right\rfloor \]

Theorem (Pikhurko, S.)

*There exists a constant \( C \) such that for all \( n \geq 5 \),*

\[ 2n - Cn^{3/4} \leq sat(n, K_{2,3}) \leq 2n - 3 \]
Theorem (Fisher, Fraughnaugh, Langley, -'95)

\[ sat(n, C_5) \leq \left\lceil \frac{10n - 10}{7} \right\rceil, \ n \neq 4 \]

Theorem (Y.C. Chen)

\[ sat(n, C_5) = \left\lceil \frac{10n - 10}{7} \right\rceil, \ n \geq 21 \]

Problem (FFL)

Determine \( sat(n, P_4 - \text{connected}) \).
Hamiltonian Cycles

Theorem

\[ \text{sat}(n, C_n) = \left\lfloor \frac{3n + 1}{2} \right\rfloor, \quad n \geq 53 \]

Bondy (’72) showed the lower bound. Clark, Entringer, Crane and Shapiro (’83-’86) gave upper bound based on Isaacs’ flower snarks (girth 5, 6). L. Stacho (’96) gave further constructions based on the Coxeter graph (girth 7).

Problem (Horák, Širáň -’86)

Is there a maximally non-hamiltonian graph of girth at least 8?
Conjecture (Bollobás - ’78)

\[ n + c_1 \frac{n}{l} \leq \text{sat}(n, C_l) \leq n + c_2 \frac{n}{l} \]

Theorem (Barefoot, Clark, Entringer, Porter, Székely, Tuza - ’96)

\[ (1 + \frac{1}{2l + 8})n \leq \text{sat}(n, C_l) \]
Theorem (Barefoot et al. - ’96)

\[
sat(n, C_l) \leq (1 + \frac{6}{l-3})n + O(l^2) \text{ for } l \text{ odd, } l \geq 9
\]

\[
sat(n, C_l) \leq (1 + \frac{4}{l-2})n + O(l^3) \text{ for } l \text{ even, } l \geq 14
\]
Theorem (Barefoot et al. - ’96)

\[\text{sat}(n, C_l) \leq (1 + \frac{1}{3} \frac{6}{l-3})n + \frac{5l^2}{4} \text{ for } l \text{ odd, } l \geq 9, \ l \geq 17, \ n \geq 7l\]

\[\text{sat}(n, C_l) \leq (1 + \frac{1}{2} \frac{4}{l-2})n + \frac{5l^2}{4} \text{ for } l \text{ even } l \geq 14, \ l \geq 10, \ n \geq 3l\]

**Theorem**

\[\text{Gould, Łuczak, S. -’06} \] For \( l = 8, 9, 11, 13 \text{ or } 15 \) and \( n \geq 2l \)

\[
\text{sat}(n, C_l) \leq \left\lfloor \frac{3n + l^2 - 9l + 15}{2} \right\rfloor
\]

\[
< \left\lfloor \frac{3n}{2} \right\rfloor + \frac{l^2}{2}
\]
Our Inspiration

Lance Armstrong’s Trek Madone SSL proto, 12/06/2004. The complete special edition Bontrager front wheel with super-minimal 19mm tubulars.
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Paths, Bipartite Graphs and General Bound
A few more questions

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Łuczak Wheel
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The Even Łuczak Wheel, $l = 2k + 2 \geq 10$
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Counting Edges of the Łuczak Wheel

For $l = 2k + 2$

$$|E(L - \text{Wheel})| = \underbrace{(n - k - a)}_{\text{Rim}} + \underbrace{\frac{n - k - a}{k}}_{\text{Spokes}} + \underbrace{a}_{\text{Flange}} + \underbrace{\binom{k}{2}}_{\text{Hub}}.$$ 

Theorem

For $k \geq 4$, $l = 2k + 2$, $n \equiv a \mod k$ and $n \geq 3l$,

$$\text{sat}(n, C_l) \leq n\left(1 + \frac{1}{k}\right) + \frac{k^2 - 3k - 2}{2} - \frac{a}{k} \leq n\left(1 + \frac{2}{l - 2}\right) + \frac{5l^2}{4}.$$
The Odd Łuczak Wheel, $l = 2k + 3 \geq 17$
Inspired by Fisher, Fraughnaugh and Langley

Present if $n - (2l - 3)$ is odd.
Recent Results and Open Problems on the Minimum Size of Saturated

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Recent Results and Open Problems on the Minimum Size of Saturated Cycles

Paths, Bipartite Graphs and General Bound

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Problem (Barefoot et al. - '96)

*Determine the value of* $l$ *which minimizes* $\text{sat}(n, C_l)$ *for fixed* $n$.

**Problem**

*Are any of these constructions optimal? Can one improve the lower bound?*
Other Subgraphs

Other values of $sat(n, F)$ known for:

- **matchings** (Mader - '73),
- **paths and stars** (Kászonyi and Tuza - '86),
- **hamiltonian path, $P_n$** (Frick and Singleton, 05; Dudek, Katona, Wojda - '06)

$$sat(n, P_n) = \left\lceil \frac{3n - 2}{2} \right\rceil, n \geq 54$$

- **longest path = detour** (Beineke, Dunbar, Frick, '05)
Difficulties and Hereditary Properties Lacking

Quote from Erdős, Hajnal and Moon:
“One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult.”

- \( \text{sat}(n, F) \not\leq \text{sat}(n + 1, F) \)
- \( \mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow \text{sat}(n, \mathcal{F}_1) \geq \text{sat}(n, \mathcal{F}_2) \)
- \( F' \subset F \not\Rightarrow \text{sat}(n, F') \leq \text{sat}(n, F) \)
Theorem (Kászonyi L. and Tuza, Z.)

Let $F$ be a graph. Set

$$u = |V(F)| - \alpha(F) - 1$$
$$s = \min\{e(F') : F' \subseteq F, \alpha(F') = \alpha(F), |V(F')| = \alpha(F) + 1\}.$$  

Then

$$\text{sat}(n, F) \leq (u + \frac{s - 1}{2})n - \frac{u(s + u)}{2}.$$  

They considered a clique on $u$ vertices joined to an $(s - 1)$-regular graph.
Best Known Lower Bound

Problem

*For an arbitrary graph $F$, determine a non-trivial lower bound on $\text{sat}(n, F)$.***
Saturation for Bipartite Graphs, $K_{s,s}$
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Saturation for Bipartite Graphs, $K_{s,s}$
Theorem (S. - '05)

For \( n \geq 3s - 3 \),

\[
\text{sat}(n, K_{s,s}) \leq \left\lfloor \frac{(3s - 3)n - (2s - 1)(s - 1)}{2} \right\rfloor - (s - 1).
\]

Theorem (S. - '05)

For \( n \geq st + s - 3 \),

\[
\text{sat}(n, K_{s}^{t}) \leq \left\lfloor \frac{(2st - s - 3)n - s^{2}t^{2} + s^{2}t + 2st - s - 1}{2} \right\rfloor - (s - 1).
\]
Theorem (Gould, S. - 06+)

For integers $t \geq 3$, $n \geq 4t - 4$,

$$sat(n, K^t_2) \leq sat(n, K^t_2, 2t - 3) = \left\lceil \frac{(4t - 5)n - 4t^2 + 6t - 1}{2} \right\rceil.$$ 

Problem

Given a fixed graph $F$, for $n$ sufficiently large determine if the function $sat(n, F, \delta)$ is monotonically increasing in $\delta$. 
And Ramsey Numbers

$F \to (F_1, \ldots, F_t)$ if any $t$ coloring of $E(F)$ contains a monochromatic $F_i$-subgraph of color $i$ for some $i \in [t]$.

**Conjecture (Hanson and Toft, ’87)**

Given $t \geq 2$ and numbers $m_i \geq 3, i \in [t]$, let

$$\mathcal{F} = \{ F : F \to (K_{m_1}, \ldots, K_{m_t}) \}.$$  

Let $r = r(K_{m_1}, \ldots, K_{m_t})$ be the classical Ramsey number. Then

$$\text{sat}(n, \mathcal{F}) = (r - 2)(n - 1) - \binom{r - 2}{2}.$$  

Problem (Pikhurko)

For even more problems see paper "Results and Open Problems on Minimum Saturated Hypergraphs", Ars Combin.

Talk and results are available online at:
http://community.middlebury.edu/~jschmitt/