Minimum Saturated Graphs & Ramsey Graphs

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joint work with
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Definition

Given a family of graphs $\mathcal{F}$, a graph $G$ is $\mathcal{F}$-saturated if

for every $F \in \mathcal{F}$, $F \not\subset G$ and

for some $F \in \mathcal{F}$, $F \subset G + e$ for any $e \in E(G)$.

Problem

*Determine the minimum number of edges of an $n$-vertex $\mathcal{F}$-saturated graph, denote this number by $\text{sat}(n, \mathcal{F})$.***
Theorem (Erdős, Hajnal, Moon - ’64)

\[ sat(n, K_k) = (k - 2)(n - 1) - \binom{k - 2}{2}, \quad n \geq k. \]

Furthermore, the only \( K_k \)-saturated graph with this many edges is \( K_{k-2} + \overline{K}_{n-k+2} \).
Subsequently, Hajnal (’65) investigated $K_k$-saturated graphs without conical vertices. Other results for $K_k$-saturated graphs with restrictions on maximum degree are given by: Hanson and Seyffarth (’84), Duffus and Hanson (’86), Erdős and Holzman (’94), Füredi and Seress (’94), and Alon, Erdős, Holzman and Krivelevich (’96).

**Theorem (Barefoot et al. - ’95)**

A $K_3$-saturated graph which is not a star must have at least $2n - 5$ edges.
Difficulties and Hereditary Properties Lacking

Quote from Erdős, Hajnal and Moon:
“One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult.”

- $sat(n, F) \nleq sat(n + 1, F)$
- $\mathcal{F}_1 \subset \mathcal{F}_2 \nRightarrow sat(n, \mathcal{F}_1) \geq sat(n, \mathcal{F}_2)$
- $F' \subset F \nRightarrow sat(n, F') \leq sat(n, F)$
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- $\text{sat}(n, F) \not\leq \text{sat}(n + 1, F)$
- $\mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow \text{sat}(n, \mathcal{F}_1) \geq \text{sat}(n, \mathcal{F}_2)$
- $F' \subset F \not\Rightarrow \text{sat}(n, F') \leq \text{sat}(n, F)$

- $\text{sat}(2k - 1, P_4) = k + 1$ and $\text{sat}(2k, P_4) = k$
- $\text{sat}(n, \{P_5, S_4\}) = n - 1 \geq \text{sat}(n, P_5)$
- $\text{sat}(n, K_4) = 2n - 3$ but $\text{sat}(n, K_5 - S_3) \leq \frac{3}{2}n$
Best known upper bound

Theorem (Kászonyi and Tuza - ’86)

Let $\mathcal{F}$ be a family of graphs. Set

$$u = u(\mathcal{F}) = \min \{|V(F)| - \alpha(F) - 1 : F \in \mathcal{F}\}$$

$$s = s(\mathcal{F}) = \min \{e(F') : F' \subseteq F \in \mathcal{F}, \alpha(F') = \alpha(F), |V(F')| = \alpha(F) + 1\}.$$

Then

$$sat(n, \mathcal{F}) \leq (u + \frac{s-1}{2})n - \frac{u(s+u)}{2}.$$ 

They considered a clique on $u$ vertices joined to an $(s-1)$-regular graph.

$$sat(n, \mathcal{F}) = O(n)$$
Best known lower bound
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A trivial lower bound:

\[ sat(n, F) \geq \frac{\delta(F)-1}{2} n \]
Best known lower bound

Problem

For an arbitrary graph $F$ determine a non-trivial lower bound on $\text{sat}(n, F)$. 
Definitions

$F \rightarrow (F_1, \ldots, F_t)$ if any $t$ coloring of $E(F)$ contains a monochromatic $F_i$-subgraph of color $i$ for some $i \in [t]$.

$F$ is $(F_1, \ldots, F_t)$-Ramsey-minimal if $F \rightarrow (F_1, \ldots, F_t)$ but for any proper subgraph $F'$ of $F$, $F' \not\rightarrow (F_1, \ldots, F_t)$.

Let $\mathcal{R}_{min}(F_1, \ldots, F_t) = \{ F : F$ is $(F_1, \ldots, F_t)$ – Ramsey – minimal$\}$. 
Main problem

Conjecture (Hanson and Toft, ’87)

Given $t \geq 2$ and numbers $m_i \geq 3, i \in [t]$.
Let $r = r(K_{m_1}, \ldots , K_{m_t})$ be the classical Ramsey number. Then

$$\text{sat}(n, R_{\text{min}}(K_{m_1}, \ldots , K_{m_t})) = (r - 2)(n - 1) - \binom{r - 2}{2}.$$
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\[
sat(n, R_{\min}(K_{m_1}, \ldots, K_{m_t})) = (r - 2)(n - 1) - \binom{r - 2}{2}.
\]

For \( t = 1 \) or \( m_2 = m_3 = \ldots m_t = 2 \), the conjecture reduces to the theorem of Erdős, Hajnal, and Moon.
Upper bound (example for $\text{sat}(n, R_{\text{min}}(K_3, K_3))$: 

![Diagram of a graph with red and blue edges]
Upper bound (example for $\text{sat}(n, R_{\text{min}}(K_3, K_3))$):

‘Clone’ a vertex.
Lower bound for \( sat(n, R_{\text{min}}(K_k, K_k)) \) follows from:

**Theorem (Burr, Erdős, Lovász - '76; Fox, Lin - '06)**

*The minimum degree of a graph in \( R_{\text{min}}(K_k, K_k) \) is at least \((k - 1)^2\).*
Lower bound for $sat(n, \mathcal{R}_{\text{min}}(K_k, K_k))$ follows from:

**Theorem (Burr, Erdős, Lovász - '76; Fox, Lin - '06)**

The minimum degree of a graph in $\mathcal{R}_{\text{min}}(K_k, K_k)$ is at least $(k - 1)^2$.

Thus, trivially,

$$sat(n, \mathcal{R}_{\text{min}}(K_k, K_k)) \geq \frac{(k - 1)^2 - 1}{2} n.$$
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Thus, trivially,

$$
\text{sat}(n, \mathcal{R}_{\min}(K_k, K_k)) \geq \frac{(k - 1)^2 - 1}{2} n.
$$

This is miserable compared to upper bound.
\( \mathcal{R}_{\text{min}}(K_3, K_3) \)-saturated graphs

\( \mathcal{R}_{\text{min}}(K_3, K_3) \)-saturated graphs were investigated by Galluccio, Simonovits, and Simonyi ('95) (using slightly different terminology) and Szabó ('96). They gave various product constructions for such graphs. These constructions generally produce graphs with ‘many’ edges.

Theorem (GSS-'95)

If \( G_1 \) and \( G_2 \) are two non-bipartite \( K_3 \)-saturated graphs, then \( G_1 + G_2 \) is a \( \mathcal{R}_{\text{min}}(K_3, K_3) \)-saturated graph.
\( R_{\text{min}}(K_3, K_3) \)-saturated graphs

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**Theorem (GSS-’95)**

If \( G_1 \) and \( G_2 \) are two non-bipartite \( K_3 \)-saturated graphs, then \( G_1 + G_2 \) is a \( R_{\text{min}}(K_3, K_3) \)-saturated graph.

**Theorem (GSS-’95)**

A \( R_{\text{min}}(K_3, K_3) \)-saturated graph has minimum degree at least 4.

This gives a slight improvement to the previous trivial lower bound for the case \( \text{sat}(n, R_{\text{min}}(K_3, K_3)) \).
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**Proof:** Apply Turán’s theorem.
Theorem (Chen, Ferrara, Gould, Magnant, S.)

For $n \geq 56$, $\text{sat}(n, R_{min}(K_3, K_3)) = 4n - 10$.

This confirms the first non-trivial case of Hanson-Toft Conjecture.
Theorem (Chen, Ferrara, Gould, Magnant, S.)

For $n \geq 11$, $\text{sat}(n, R_{min}(K_3, P_3)) = \left\lfloor \frac{5n}{2} \right\rfloor - 5$. 
A natural upper bound for $\text{sat}(n, R_{\min}(K_3, P_3))$

For a natural upper bound, one might think of Chvátal’s ‘clique vs. tree’ Theorem for the case $r(K_3, P_3)$. 
A natural upper bound for \( \text{sat}(n, \mathcal{R}_{min}(K_3, P_3)) \)

For a natural upper bound, one might think of Chvátal’s ‘clique vs. tree’ Theorem for the case \( r(K_3, P_3) \) and do some ‘cloning’.
A fact for $\mathcal{R}_{min}(K_3, P_3)$-saturated graphs

**Fact:** In any good red/blue-coloring of a $\mathcal{R}_{min}(K_3, P_3)$-saturated graph any edge lying in three or more triangles must be colored blue (and so the other edges in the triangles must be colored red).
A better (and a best) upper bound for \( \text{sat}(n, R_{\text{min}}(K_3, P_3)) \)
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So the coloring is unique, and the graph is $R_{\min}(K_3, P_3)$-saturated. This provides the upper bound
A proof of the lower bound

Let $G$ be an $n$-vertex $\mathcal{R}_{\text{min}}(K_3, P_3)$-saturated graph with minimum number of edges. Consider a good coloring of $G$ with maximum number of red edges. Let $G_b$ denote the blue graph and $G_r$ the red graph.
A proof of the lower bound

Let $G$ be an $n$-vertex $\mathcal{R}_{\min}(K_3, P_3)$-saturated graph with minimum number of edges. Consider a good coloring of $G$ with maximum number of red edges. Let $G_b$ denote the blue graph and $G_r$ the red graph.

**Fact:** $G_b$ is a matching the edges of which are incident with at least $n - 2$ vertices.
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Let $G$ be an $n$-vertex $\mathcal{R}_{\text{min}}(K_3, P_3)$-saturated graph with minimum number of edges. Consider a good coloring of $G$ with maximum number of red edges. Let $G_b$ denote the blue graph and $G_r$ the red graph.

**Fact:** $G_b$ is a matching the edges of which are incident with at least $n - 2$ vertices.

**Claim:** $G_r$ is 2-connected.
A proof of the lower bound

Claim: $G_r$ is 2-connected.

- Obviously $G_r$ is connected as otherwise $G_b$ would contain a complete bipartite graph - a contradiction.
- Suppose $G_r$ has connectivity 1.
**Claim:** \( G_r \) is 2-connected.
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We see that $G$ is not $R_{\text{min}}(K_3, P_3)$-saturated. Establishes claim that $G_r$ is 2-connected.
Completing the proof

Case: $n$ is odd

$G_r$ is $K_3$-saturated — if not, either add a red edge to $G$ or re-color blue edge of $G$ red. Apply a theorem of Barefoot et al. ('95) to $G_r$ to obtain that $G_r$ has at least $2n - 5$ edges. Also, $G_b$ must have $\lfloor \frac{n}{2} \rfloor$ edges. Thus, $G$ has at least $\lfloor \frac{5n}{2} \rfloor - 5$ edges.
Completing the proof

**Case: \( n \) is odd**

\( G_r \) is \( K_3 \)-saturated — if not, either add a red edge to \( G \) or re-color blue edge of \( G \) red. Apply a theorem of Barefoot et al. ('95) to \( G_r \) to obtain that \( G_r \) has at least \( 2n - 5 \) edges. Also, \( G_b \) must have \( \left\lfloor \frac{n}{2} \right\rfloor \) edges. Thus, \( G \) has at least \( \left\lfloor \frac{5n}{2} \right\rfloor - 5 \) edges.

**Case: \( n \) is even**

We omit here.

\( \square \)
A few words about the proof of $\text{sat}(n, R_{\text{min}}(K_3, K_3)) = 4n - 10$. 
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A few words about the conjecture.
Open problems and questions:

- Hanson-Toft conjecture remains open in general.
- Does every $R_{min}(K_3, K_3)$-saturated graph contain a $K_4$? [GSS-’95]
- If $G$ is a $R_{min}(K_3, K_3)$-saturated graph containing a $K_5$ must it also contain a $K_6 - e$? [GSS-’95]
- Can one find a finite set $Q_1, Q_2, \ldots, Q_m$ of $R_{min}(K_3, K_3)$-saturated graphs so that every $R_{min}(K_3, K_3)$-saturated graph contains at least one of them? [GSS-’95]
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Thanks!