On A Relationship of Two Extremal Functions

John Schmitt

Middlebury College
joint work with M. Ferrara, R. Gould, T. Łuczak, O. Pikhurko

July 2006
Horizon of Combinatorics

Definition

A non-negative, non-increasing, integer sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be **graphic** if there exists a graph G with π as its degree sequence.

Definition

For a given subgraph F, a sequence π is **potentially F-graphic** if there is **some** realization of π containing F as a subgraph.

π is said to be **forcibly F-graphic** if **every** realization of π contains F as a subgraph.

Let $\Sigma d_i = d_1 + d_2 + \cdots + d_n$.

Turán's problem rephrased:

Given a subgraph F, determine the least even integer m s.t. $\Sigma d_i \geq m \Rightarrow \pi$ is forcibly F-graphic.

Problem

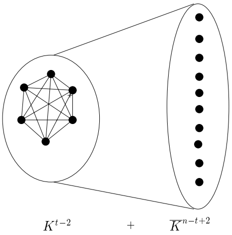
Given a subgraph F, determine the least even integer m s.t. $\Sigma d_i \geq m \Rightarrow \pi$ is potentially F-graphic.

Denote m by $\sigma(F, n)$.

Conjecture

(EJL - 1991) For n sufficiently large,
$\sigma(K^t, n) = (t-2)(2n-t+1)+2.$

Lower bound arises from considering:



$$\pi = ((n-1)^{t-2}, (t-2)^{n-t+2})$$

Erdős, Jacobson, Lehel Conjecture

Cases settled:

- $t = 3$ Erdős, Jacobson, & Lehel(1991),
- $t = 4$ Gould, Jacobson, & Lehel(1999), Li & Song(1998),
- $t = 5$ Li & Song(1998),
- $t \geq 6$ Li, Song, & Luo(1998)
- $t \geq 3$ Ferrara, Gould, S.(2005) - purely graph-theoretic proof.

Theorem

For n sufficiently large, $\sigma(K^t, n) = (t-2)(2n-t+1)+2.$

Definition

A graph G is **F-saturated** if:

$F \not\subset G$

$F \subset G + e$ for any $e \in E(\overline{G})$

Problem

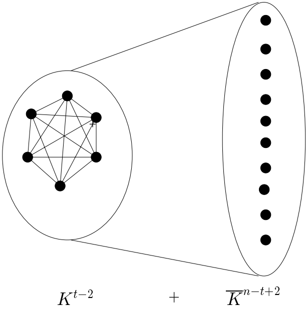
Determine the minimum number of edges, $sat(n, F)$, of an F-saturated graph.

History

Theorem (Erdős, Hajnal, Moon - 1964)

$$sat(n, K^t) = (t-2)(n-1) - \binom{t-2}{2}$$

Furthermore, the only K^t-saturated graph with this many edges is $K^{t-2} + \overline{K}^{n-t+2}$.



Other Subgraphs

Exact values of $sat(n, F)$ known for:

- $K_{2,2}$(Ollmann -'72; Tuza '86)
- **matchings** (Mader - '73),
- **paths and stars** (Kászonyi and Tuza - '86),
- **hamiltonian cycle,** C_n (Bondy - '72; Clark et al. '86-'92)

$$sat(n, C_n) = \lfloor \frac{3n+1}{2} \rfloor, n \geq 53.$$

Other Subgraphs

Recent progress made on:

- **five cycle,** C_5 (Y.C. Chen '06+)

$$sat(n, C_5) = \lceil \frac{10n - 10}{7} \rceil, n \geq 21$$

- **hamiltonian path,** P_n (Frick and Singleton, 05)

$$sat(n, P_n) = \lceil \frac{3n - 2}{2} \rceil, n \geq 54$$

- **longest path = detour**(Beineke, Dunbar, Frick, '05)
- $K_{2,3}$(Pikhurko, S.)

$$2n - Cn^{3/4} \leq sat(n, K_{2,3}) \leq 2n - 3$$

- $K_{t(2)}$ (Gould, S. -'06)

Saturation for Cycles

Theorem (Barefoot et.al. - '96)

[Gould, Łuczak, S. - '06]

$sat(n, C_l) \leq (1 + \frac{1}{3}\frac{6}{l-3})n + \frac{5l^2}{4}$ *for* l *odd,* $l \geq 9$, $l \geq 17, n \geq 7l$

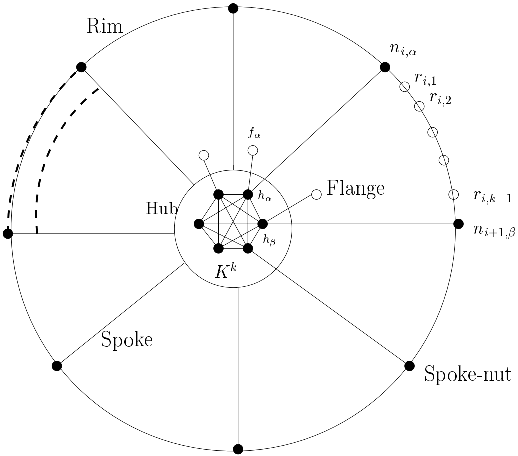
$sat(n, C_l) \leq (1 + \frac{1}{2}\frac{4}{l-2})n + \frac{5l^2}{4}$ *for* l *even* $l \geq 14$, $l \geq 10, n \geq 3l$

Theorem

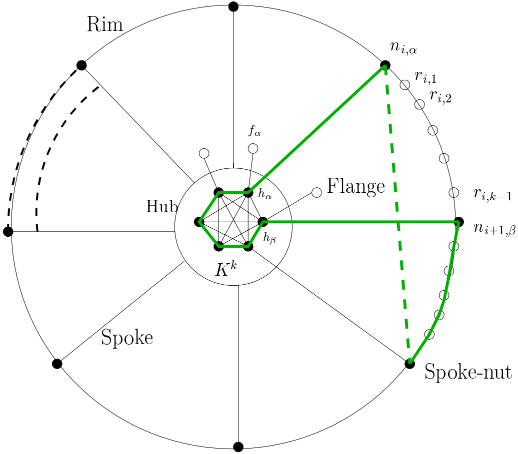
[Gould, Łuczak, S. -'06] For $l = 8, 9, 11, 13$ *or* 15 *and* $n \geq 2l$

$$\begin{aligned} sat(n, C_l) &\leq \left\lceil \frac{3n + l^2 - 9l + 15}{2} \right\rceil \\ &< \left\lceil \frac{3n}{2} \right\rceil + \frac{l^2}{2} \end{aligned}$$

Łuczak Wheel - A C_{2k+2} Saturated Graph



Łuczak Wheel - A C_{2k+2} Saturated Graph



Hereditary Properties Lacking

The function $sat(n, F)$ is a slippery fish:

- $sat(n, F) \not\leq sat(n+1, F)$
- $\mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow sat(n, \mathcal{F}_1) \geq sat(n, \mathcal{F}_2)$
- $F' \subset F \not\Rightarrow sat(n, F') \leq sat(n, F)$

Best known upper bound

Theorem (Kászonyi and Tuza - '86)

Let F be a graph. Set

$$u := u(F) = |V(F)| - \alpha(F) - 1$$

$$s := s(F) = min|\{e(H) : H \subset F, \alpha(H) = \alpha(F), |H| = \alpha(F) + 1\}|.$$

Then

$$sat(n, F) \leq (u + \frac{s-1}{2})n - \frac{u(s+u)}{2}.$$

Hereditary Properties of $\sigma(F, n)$

Let $\mathcal{F} = \{F_1, F_2, \ldots\}$ be a family of graphs.

- $\sigma(\mathcal{F}, n) \leq \sigma(\mathcal{F}, n+1)$ for every n and $\mathcal{F}$.
- If $\mathcal{F}_1 \subset \mathcal{F}_2$ then $\sigma(\mathcal{F}_1, n) \geq \sigma(\mathcal{F}_2, n)$ for every n.
- If F is a subgraph of F' then $\sigma(F, n) \leq \sigma(F', n)$ for every n.

This last property:

- together with the Theorem for cliques gives an upper bound for $\sigma(F, n)$,
- is useful in proving a lower bound for $\sigma(F, n)$.

A Lower Bound for $\sigma(F, n)$

As before set $u := u(F) = |V(F)| - \alpha(F) - 1$, and define

$$d := d(F) = min\{\Delta(H) : H \subset F, |H| = \alpha(F) + 1\}.$$

Consider the following sequence,

$$\pi(F, n) = ((n-1)^u, (u+d-1)^{n-u}).$$

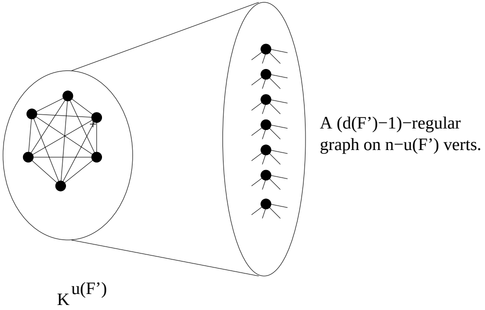
Proposition (Ferrara, S.)

Given a graph F and n sufficiently large then,

$$\sigma(F, n) \geq max\{n(2u(F') + d(F') - 1) | F' \subseteq F\}. \tag{1}$$

Proof of Lower Bound

PROOF: Let $F' \subseteq F$ be the subgraph which achieves the max. Consider,



$$u(F') = |V(F')| - \alpha(F') - 1$$
$$d(F') = min\{\Delta(H) : H \subset F', |H| = \alpha(F') + 1\}$$

The Inequality

Theorem of Kászonyi and Tuza and this proposition immediately imply:

Theorem (Ferrara, S.)

For a given subgraph F, if there exists an $F' \subseteq F$ with

$$2u(F') + d(F') - 1 \geq 2u(F) + s(F) - 1$$

then for n sufficiently large we have

$$2sat(n, F) < \sigma(F, n).$$

Conjecture

The above inequality holds for all graphs.