# The Efficiency of the Bicycle Wheel

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Joint work with:

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Definition

A graph $G$ is **$C_l$-saturated** if:

$C_l \not\subset G$

$C_l \subset G + e$ for any $e \in E(\overline{G})$

Problem

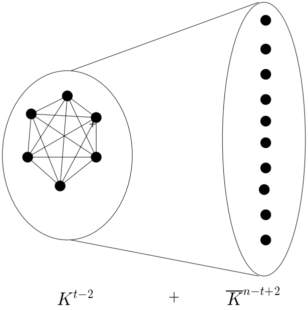
*Determine the minimum number of edges,* $sat(n, C_l)$*, of an* $C_l$*-saturated graph.*

# History

Theorem (Erdős, Hajnal, Moon - 1963)

$$sat(n, K^t) = (t-2)(n-1) - \binom{t-2}{2}$$

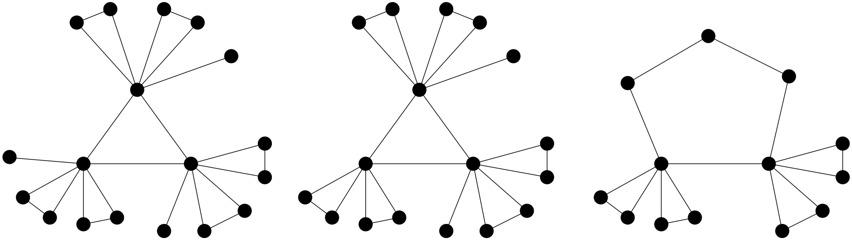
*Furthermore, the only $K^t$-saturated graph with this many edges is $K^{t-2} + \overline{K}^{n-t+2}$.*



# History

Theorem[Ollmann - '72, Tuza - '86]

$$sat(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor, \quad n \geq 5$$



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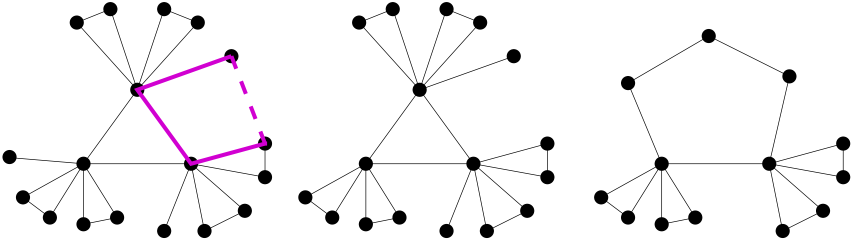

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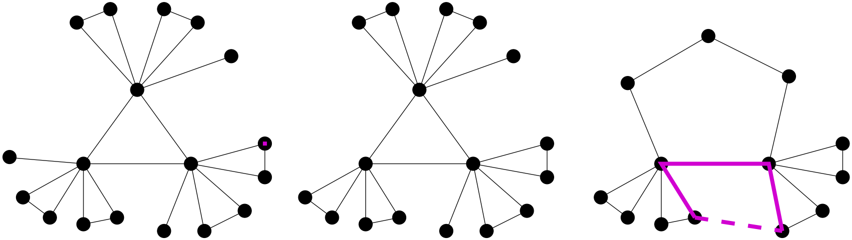


# Other Cycles

Exact values of $sat(n, C_l)$ known for:

- **five cycle,** $C_5$ (Y.C. Chen '05+)

$$sat(n, C_5) = \lceil \frac{10n - 10}{7} \rceil, n \geq 21$$

- **hamiltonian cycle,** $C_n$ (Clark et al. '86-'92)

$$sat(n, C_n) = \lfloor \frac{3n + 1}{2} \rfloor, n \geq 53.$$

# Difficulties and Hereditary Properties Lacking

Quote from Erdős, Hajnal and Moon:
"One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult."

- $sat(n, F) \not\leq sat(n+1, F)$
- $\mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow sat(n, \mathcal{F}_1) \geq sat(n, \mathcal{F}_2)$
- $F' \subset F \not\Rightarrow sat(n, F') \leq sat(n, F)$

# Saturation for Cycles

- **Conjecture (Bollobás - '78)**

$$n + c_1\frac{n}{l} \leq sat(n, C_l) \leq n + c_2\frac{n}{l}$$

- Theorem (Barefoot *et al.* - '96)

$$(1 + \frac{1}{2l+8})n \leq sat(n, C_l)$$

# Saturation for Cycles

**Theorem (Barefoot *et al.* - '96)**

$sat(n, C_l) \leq (1 + \frac{6}{l-3})n + O(l^2)$ *for $l$ odd,* $l \geq 9$

$sat(n, C_l) \leq (1 + \frac{4}{l-2})n + O(l^3)$ *for $l$ even,* $l \geq 14$

# Saturation for Cycles

Theorem (Barefoot *et al.* - '96)

*[Gould, Łuczak, S.]*

$sat(n, C_l) \leq (1 + \frac{1}{3}\frac{6}{l-3})n + \frac{5l^2}{4}$ *for l odd,* $l \geq 9$, $l \geq 17, n \geq 7l$

$sat(n, C_l) \leq (1 + \frac{1}{2}\frac{4}{l-2})n + \frac{5l^2}{4}$ *for l even* $l \geq 14$, $l \geq 10, n \geq 3l$

Theorem

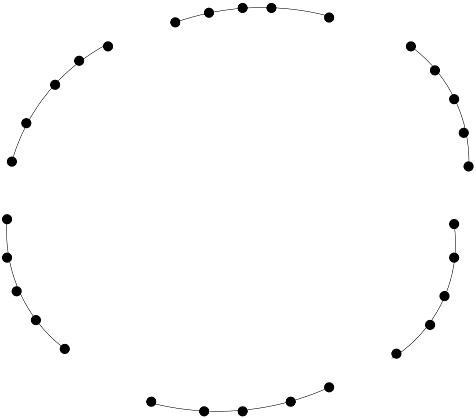
*[Gould, Łuczak, S.] For* $l = 8, 9, 11, 13$ *or* $15$ *and* $n \geq 2l$

$$\begin{aligned} sat(n, C_l) &\leq \left\lceil \frac{3n + l^2 - 9l + 15}{2} \right\rceil \\ &< \left\lceil \frac{3n}{2} \right\rceil + \frac{l^2}{2} \end{aligned}$$

# Our Inspiration

Lance Armstrong's Trek Madone SSL proto, 12/06/2004. The complete special edition Bontrager front wheel with super-minimal 19mm tubulars.

# Logic of First Construction



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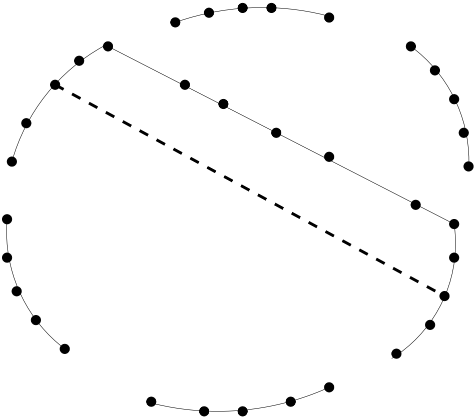

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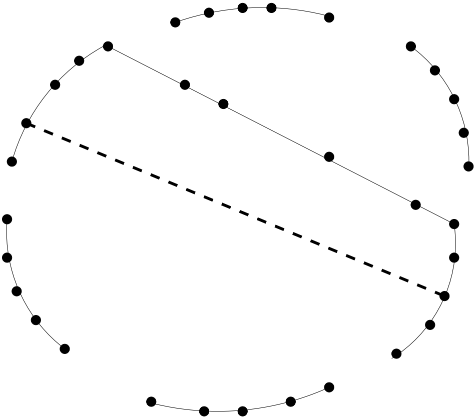
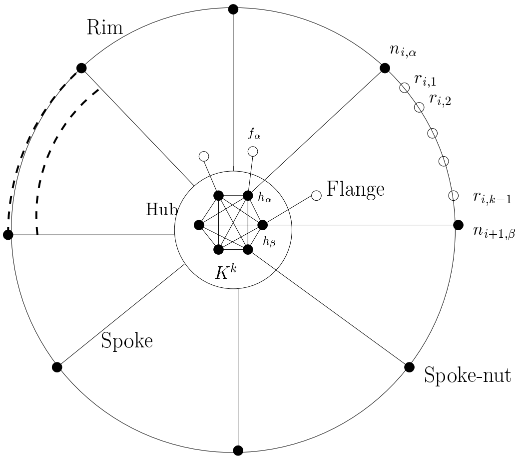


# The Even Łuczak Wheel, $l = 2k + 2 \geq 10$



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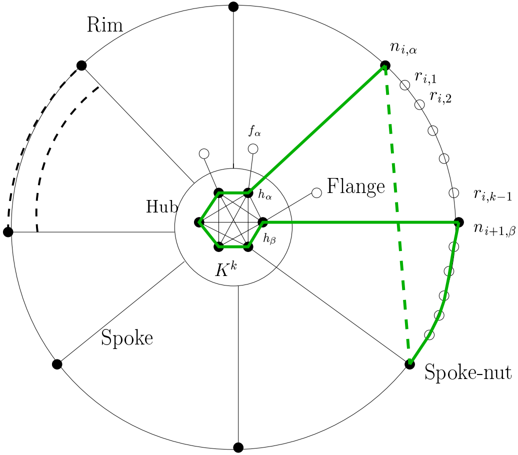

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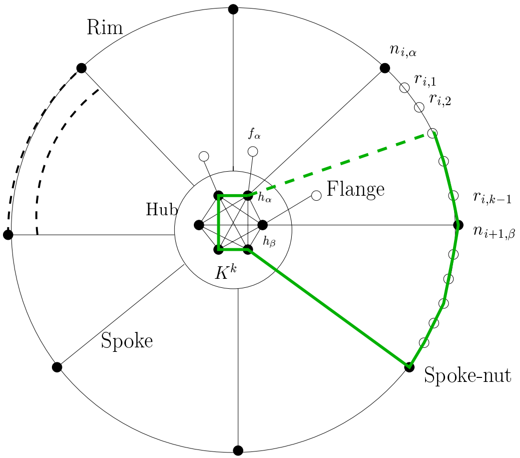

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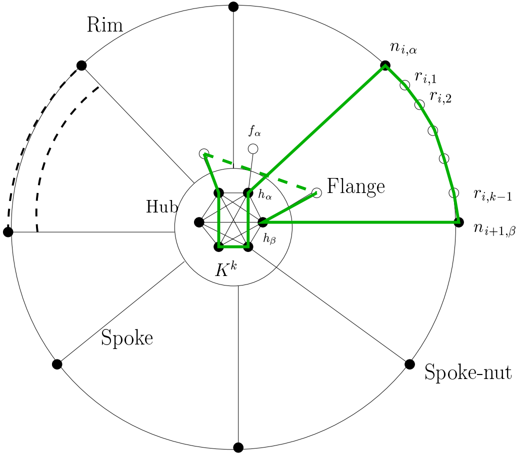


# Counting Edges of the Łuczak Wheel

For $l = 2k + 2$

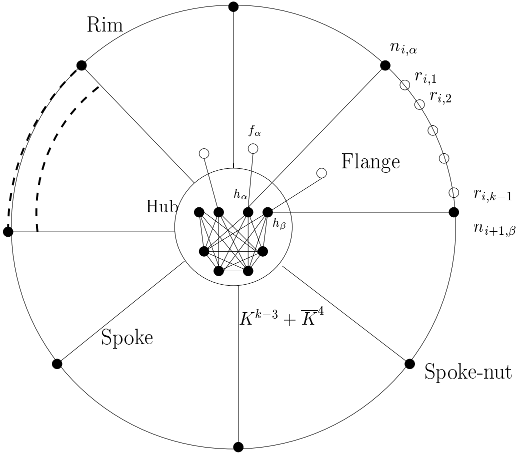
$$|E(L - Wheel)| \quad = \quad \overbrace{(n-k-a)}^{Rim} + \overbrace{\frac{n-k-a}{k}}^{Spokes} + \overbrace{a + \Sigma_{i=1}^{k}\binom{a_i}{2}}^{Flange} + \overbrace{\binom{k}{2}}^{Hub}.$$

**Theorem**

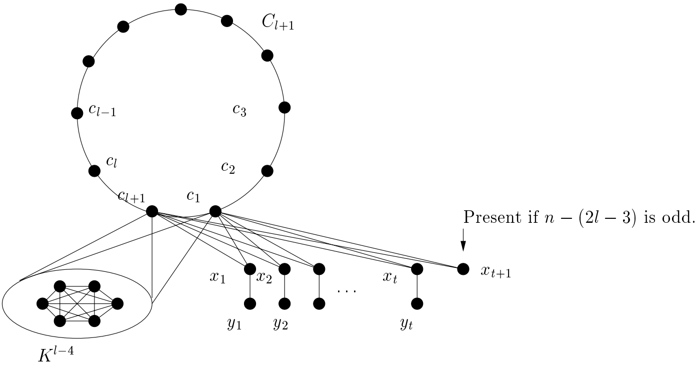
*For $k \geq 4$, $l = 2k+2$, $n \equiv a \mod k$ and $n \geq 3l$,*

$$\begin{aligned} sat(n, C_l) &\leq n(1 + \frac{1}{k}) + \frac{k^2 - 3k - 2}{2} - \frac{a}{k} + \Sigma_{i=1}^{k}\binom{a_i}{2} \\ &\leq n(1 + \frac{2}{l-2}) + \frac{5l^2}{4}. \end{aligned}$$

# The Odd Łuczak Wheel, $l = 2k + 3 \geq 17$



# Another Construction, $l \geq 5$



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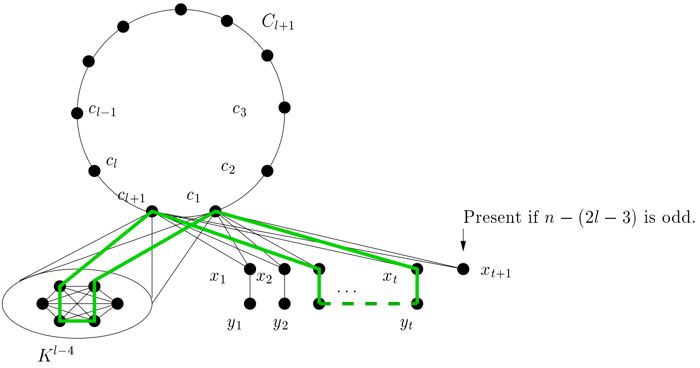

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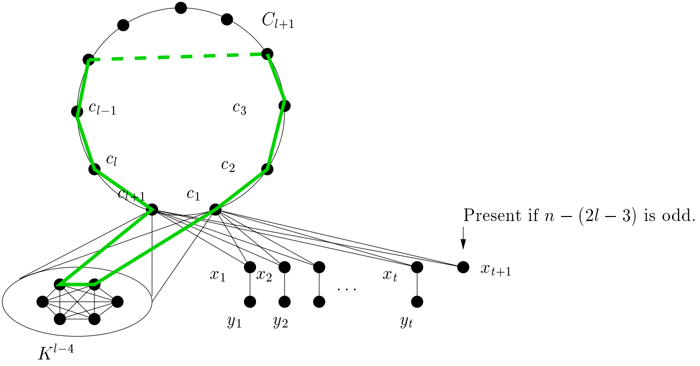

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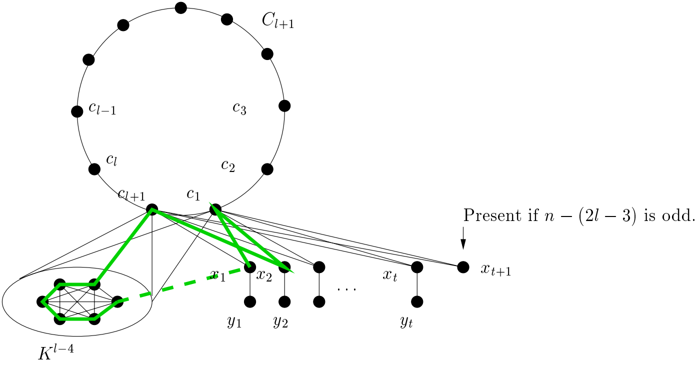

# Summary of Results on Cycles

| $C_l$-saturated graphs of minimum size | | | |
|---|---|---|---|
| $l$ | $sat(n, C_l)$ | $n \geq$ | Reference |
| 3 | $= n - 1$ | 3 | Erdős et al. |
| 4 | $\lfloor \frac{3n-5}{2} \rfloor$ | 5 | Ollmann, Tuza |
| 5 | $\lceil \frac{10n-10}{7} \rceil$ | 21 | Chen |
| 6 | $\leq \frac{3n}{2}$ | 11 | Barefoot et al. |
| 7 | $\leq \frac{7n+12}{5}$ | 10 | Barefoot et al. |
| 8,9,11,13,15 | $\leq \frac{3n}{2} + \frac{l^2}{2}$ | $2l$ | "Other" constructi |
| $\geq 10$ and $\equiv 0 \mod 2$ | $\leq \left(1 + \frac{2}{l-2}\right) n + \frac{5l^2}{4}$ | $3l$ | Even Wheel |
| $\geq 17$ and $\equiv 1 \mod 2$ | $\leq \left(1 + \frac{2}{l-3}\right) n + \frac{5l^2}{4}$ | $7l$ | Odd Wheel |
| n | $\lfloor \frac{3n+1}{2} \rfloor$ | 20 | Clark et al. |

Table: A Summary of Results for $sat(n, C_l)$

# A few questions

- Problem
  *Are any of these constructions optimal? Can one improve the lower bounds?*
- Problem (Barefoot et.al. - '96)
  *Determine the value of $l$ which minimizes $sat(n, C_l)$ for fixed $n$.*

Barefoot, C.A., Clark, L.H., Entringer, R.C., Porter, T.D., Székely, L.A., Tuza, Zs. Cycle-saturated graphs of minimum size, Discrete Mathematics 150 (1996) 31-48.

Bollobás, B., Extremal Graph Theory, Academic Press Inc. (1978).

Bondy, J.A. Variations on the hamiltonian theme, Canad. Math. Bull. 15 (1972) 57-62.

Chen, Y.C., personal communication

Clark, L.H., Crane, R.P., Entringer, R.C., Shapiro, H.D., On smallest maximally nonhamiltonian graphs, Congress. Numer. 53 (1986) 215-220.

Clark, L.H., Entringer, R.C., Smallest maximally nonhamiltonian graphs, Period. Math. Hungar. 15 (1983) 57-68.

Clark, L.H., Entringer, R.C., Shapiro, H.D., Smallest maximally nonhamiltonian graphs II, Graphs Combin. 8 (1992) 225-231.

- Erdős, P., Hajnal, A. and Moon, J.W., A problem in graph theory, Amer. Math. Monthly 71 (1964)1107-1110.
- Kászonyi, L. and Tuza, Z., Saturated graphs with minimal number of edges, J. Graph Theory 10 (1986) 203-210.
- Ollmann, L.T., $K_{2,2}$-saturated graphs with a minimal number of edges, in Proc. 3rd SouthEast Conference on Combinatorics, Graph Theory and Computing, (1972) 367-392.
- Tuza, Z., $C_4$-saturated graphs of minimum size, Acta Universitatis Carolinae - Mathematica et Physica, Vol. 30 No. 2 (1989) 161-167.