An Erdős-Stone Type Conjecture

John Schmitt

Middlebury College
joint work with Michael Ferrara (U. Colorado, Denver)

July, 2006
6th Czech Slovak International Symposium

Fundamental Results in Extremal Graph Theory

Question

What is the maximum number of edges, $ex(n, F)$, of an F-free graph on n vertices?

Theorem

(Turán - 1941)

$$ex(n, K^t) \sim \frac{t-2}{t-1} n^2.$$

Theorem

(Erdős-Stone-Simonovits - 1966)

$$ex(n, F) \sim \left(1 - \frac{1}{\chi(F) - 1}\right) n^2.$$

Definition

A non-negative, non-increasing, integer sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be **graphic** if there exists a graph G with π as its degree sequence.

Definition

For a given subgraph F, a sequence π is **potentially F-graphic** if there is **some** realization of π containing F as a subgraph.

π is said to be **forcibly F-graphic** if **every** realization of π contains F as a subgraph.

Let $\Sigma d_i = d_1 + d_2 + \cdots + d_n = \sigma(\pi)$.

Turán's problem rephrased:

Given a subgraph F, determine the least even integer m s.t. $\Sigma d_i \geq m \Rightarrow \pi$ is forcibly F-graphic.

Problem

Given a subgraph F, determine the least even integer m s.t. $\Sigma d_i \geq m \Rightarrow \pi$ is potentially F-graphic.

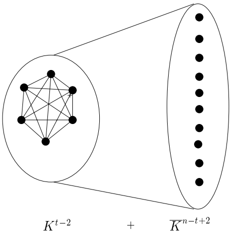
Denote m by $\sigma(F, n)$.

Erdős, Jacobson, Lehel Conjecture

Conjecture

(EJL - 1991) For n sufficiently large,
$\sigma(K^t, n) = (t-2)(2n-t+1)+2.$

Lower bound arises from considering:



$\pi = ((n-1)^{t-2}, (t-2)^{n-t+2})$

Erdős, Jacobson, Lehel Conjecture

Conjecture settled:

- $t = 3$ Erdős, Jacobson, & Lehel(1991),
- $t = 4$ Gould, Jacobson, & Lehel(1999), Li & Song(1998),
- $t = 5$ Li & Song(1998),
- $t \geq 6$ Li, Song, & Luo(1998)
- $t \geq 3$ Ferrara, Gould, S.(2005) - purely graph-theoretic proof.

Theorem

For n sufficiently large, $\sigma(K^t, n) = (t-2)(2n-t+1)+2$.

A General Lower Bound - The Set-up

Let $\alpha(F)$ denote the independence number of F and define:

$$u := u(F) = |V(F)| - \alpha(F) - 1,$$

and

$$d := d(F) = min\{\Delta(H) : H \subset F, |H| = \alpha(F) + 1\}.$$

Consider the following sequence,

$$\pi(F, n) = ((n-1)^u, (u + d - 1)^{n-u}).$$

A General Lower Bound

If F' is a subgraph of F then $\sigma(F', n) \leq \sigma(F, n)$ for every n.

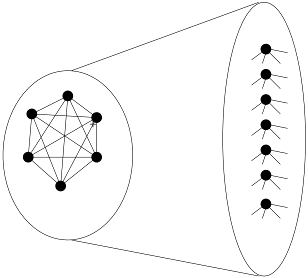
Proposition (Ferrara, S.)

Given a graph F and n sufficiently large then,

$$\begin{aligned} \sigma(F, n) &\geq max\{\sigma(\pi(F', n)) + 2 | F' \subseteq F\} & (1) \\ &\sim max\{n(2u(F') + d(F') - 1) | F' \subseteq F\} & (2) \end{aligned}$$

Proof of Lower Bound

PROOF: Let $F' \subseteq F$ be the subgraph which achieves the max. Consider,



$$u(F') = |V(F')| - \alpha(F') - 1$$
$$d(F') = min\{\Delta(H) : H \subset F', |H| = \alpha(F') + 1\}$$

A Stronger Lower Bound

Let $v_i(H)$ be the number of vertices of degree i in H. Let $S_i(H)$ denote the set of induced subgraphs on $\alpha + 1$ vertices with $v_i(H) > 0$.

For all i, $d \leq i \leq \alpha - 1$ define:

$m_i = min_{S_i(H)}\{\text{vertices of degree at least i}\}$

$n_d = m_d - 1$ and $n_i = min\{m_i - 1, n_{i-1}\}$

Finally, set $\delta_{\alpha-1} = n_{\alpha-1}$ and for all i, $d \leq i \leq \alpha - 2$ define $\delta_i = n_i - n_{i+1}$

$$\begin{aligned}\pi^*(F,n) &= ((n-1)^u, (u+\alpha-1)^{\delta_{\alpha-1}}, (u+\alpha-2)^{\delta_{\alpha-2}}, \ldots \\ &\quad (u+d)^{\delta_d}, (u+d-1)^{n-u-\Sigma\delta_i}).\end{aligned}$$

An Example

Let $F = K_{6,6}$.

Then $u(K_{6,6}) = 12 - 6 - 1 = 5$ and $d(K_{6,6}) = 4$.

$m_4 = 3$ and $m_5 = 2$

$n_4 = m_4 - 1 = 2$ and $n_5 = min\{m_5 - 1, n_4\} = 1$

$\delta_5 = n_5 = 1$ and $\delta_4 = n_5 - n_4 = 1$

Thus,

$$\pi^*(K_{6,6}, n) = ((n-1)^5, 10, 9, 8^{n-7})$$

A Stronger Lower Bound

Theorem (Ferrara, S.)

Given a graph F and n sufficiently large then,

$$\sigma(F, n) \geq max\{\sigma(\pi^*(F', n)) + 2 | F' \subseteq F\}$$

When Does Equality Hold?

- **cliques**
- **complete bipartite graphs** Chen, Li, Yin '04; Gould, Jacobson, Lehel '99; Li, Yin '02
- **complete multipartite graphs** G. Chen, Ferrara, Gould, S. - sub.; Ferrara, Gould, S. - sub
- **matchings** Gould, Jacobson, Lehel '99
- **cycles** Lai '04
- **friendship graph** Ferrara, Gould, S. '06
- **split graphs** Chen, Yin - sub.
- **clique minus an edge** Lai '01; Li, Mao, Yin '05
- **disjoint union of cliques** Ferrara - sub.

Our Conjecture

Conjecture

Given a graph F and n sufficiently large then,

$$\sigma(F,n) = max\{\sigma(\pi^*(F',n)) + 2 | F' \subseteq F\}$$

Conjecture

(weak) Given a graph F, let $\epsilon > 0$. Then there exists an $n_0 = n_0(\epsilon, F)$ such that for any $n > n_0$

$$\sigma(F,n) \leq max\{(n(2u(F') + d(F') - 1 + \epsilon) | F' \subseteq F\}.$$