A Lower Bound for Potentially F-Graphic Degree Sequences

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Question

What is the maximum number of edges, $ex(n, F)$, of an F-free graph on n vertices?

Theorem

(Turán - 1941)

$$ex(n, K^t) \sim \frac{t-2}{t-1} n^2.$$

Theorem

(Erdős-Stone-Simonovits - 1966)

$$ex(n, F) \sim (1 - \frac{1}{\chi(F) - 1}) n^2.$$

Definition

A non-negative, non-increasing, integer sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be **graphic** if there exists a graph G with π as its degree sequence.

Definition

For a given subgraph F, a sequence π is **potentially F-graphic** if there is **some** realization of π containing F as a subgraph.

π is said to be **forcibly F-graphic** if **every** realization of π contains F as a subgraph.

Let $\Sigma d_i = d_1 + d_2 + \cdots + d_n = \sigma(\pi)$.

Turán's problem rephrased:

Given a subgraph F, determine the least even integer m s.t. $\Sigma d_i \geq m \Rightarrow \pi$ is forcibly F-graphic.

Problem

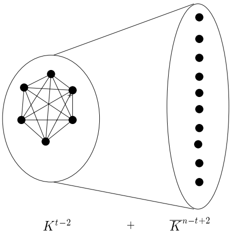
Given a subgraph F, determine the least even integer m s.t. $\Sigma d_i \geq m \Rightarrow \pi$ is potentially F-graphic.

Denote m by $\sigma(F, n)$.

Conjecture

(EJL - 1991) For n sufficiently large,
$\sigma(K^t, n) = (t-2)(2n-t+1)+2.$

Lower bound arises from considering:



$\pi = ((n-1)^{t-2}, (t-2)^{n-t+2})$

Erdős, Jacobson, Lehel Conjecture

Conjecture settled:

- $t = 3$ Erdős, Jacobson, & Lehel(1991),
- $t = 4$ Gould, Jacobson, & Lehel(1999), Li & Song(1998),
- $t = 5$ Li & Song(1998),
- $t \geq 6$ Li, Song, & Luo(1998)
- $t \geq 3$ Ferrara, Gould, S.(2005) - purely graph-theoretic proof.

Theorem

For n sufficiently large, $\sigma(K^t, n) = (t-2)(2n-t+1)+2.$

A General Lower Bound - The Set-up

Let $\alpha(F)$ denote the independence number of F and define:

$$u := u(F) = |V(F)| - \alpha(F) - 1,$$

and

$$d := d(F) = min\{\Delta(H) : H \subset F, |H| = \alpha(F) + 1\}.$$

Consider the following sequence,

$$\pi(F, n) = ((n-1)^u, (u+d-1)^{n-u}).$$

A General Lower Bound

If F' is a subgraph of F then $\sigma(F', n) \leq \sigma(F, n)$ for every n.

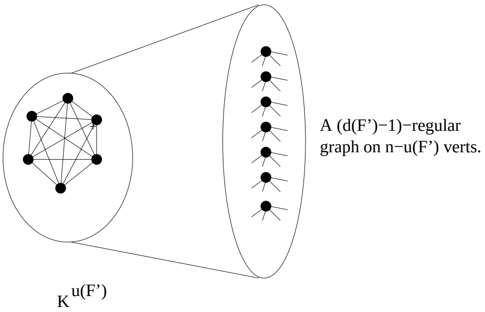
Proposition (Ferrara, S.)

Given a graph F and n sufficiently large then,

$$\begin{aligned} \sigma(F, n) &\geq max\{\sigma(\pi(F', n)) + 2 | F' \subseteq F\} && (1) \\ &\sim max\{n(2u(F') + d(F') - 1) | F' \subseteq F\} && (2) \end{aligned}$$

Proof of Lower Bound

PROOF: Let $F' \subseteq F$ be the subgraph which achieves the max. Consider,



$$u(F') = |V(F')| - \alpha(F') - 1$$
$$d(F') = min\{\Delta(H) : H \subset F', |H| = \alpha(F') + 1\}$$

A Stronger Lower Bound

Let $v_i(H)$ is the number of vertices of degree i in H. Let $S_i(H)$ denote the set of induced subgraphs on $\alpha + 1$ vertices with $v_i(H) > 0$.

For all $i, \; d \leq i \leq \alpha - 1$ define:

$m_i = min_{S_i(H)}\{\text{vertices of degree at leasti}\}$

$n_d = m_d - 1$ and $n_i = min\{m_i - 1, n_{i-1}\}$

Finally, set $\delta_{\alpha-1} = n_{\alpha-1}$ and for all $i, \; d \leq i \leq \alpha - 2$ define $\delta_i = n_i - n_{i+1}$

$$\begin{aligned} \pi^*(F, n) \quad = \quad & ((n-1)^u, (u+\alpha-1)^{\delta_{\alpha-1}}, (u+\alpha-2)^{\delta_{\alpha-2}}, \ldots \\ & (u+d)^{\delta_d}, (u+d-1)^{n-u-\Sigma\delta_i}). \end{aligned}$$

An Example

Let $F = K_{6,6}$.

Then $u(K_{6,6}) = 12 - 6 - 1 = 5$ and $d(K_{6,6}) = 4$.

$m_4 = 3$ and $m_5 = 2$

$n_4 = m_4 - 1 = 2$ and $n_5 = min\{m_5 - 1, n_4\} = 1$

$\delta_5 = n_5 = 1$ and $\delta_4 = n_5 - n_4 = 1$

Thus,

$$\pi^*(K_{6,6}, n) = ((n-1)^5, 10, 9, 8^{n-7})$$

A Stronger Lower Bound

Theorem (Ferrara, S.)

Given a graph F and n sufficiently large then,

$$\sigma(F, n) \geq max\{\sigma(\pi^*(F', n)) + 2 | F' \subseteq F\}$$

When Does Equality Hold?

- **cliques**
- **complete bipartite graphs** Chen, Li, Yin '04; Gould, Jacobson, Lehel '99; Li, Yin '02
- **complete multipartite graphs** G. Chen, Ferrara, Gould, S. - sub.; Ferrara, Gould, S. - sub
- **matchings** Gould, Jacobson, Lehel '99
- **cycles** Lai '04
- **friendship graph** Ferrara, Gould, S. '06
- **split graphs** Chen, Yin - sub.
- **clique minus an edge** Lai '01; Li, Mao, Yin '05
- **disjoint union of cliques** Ferrara - sub.

Our Conjecture

Conjecture

Given a graph F and n sufficiently large then,

$$\sigma(F,n) = max\{\sigma(\pi^*(F',n)) + 2|F' \subseteq F\}$$

Conjecture

(weak) Given a graph F, let $\epsilon > 0$. Then there exists an $n_0 = n_0(\epsilon, F)$ such that for any $n > n_0$

$$\sigma(F,n) \leq max\{(n(2u(F') + d(F') - 1 + \epsilon)|F' \subseteq F\}.$$