Generalizing the degree sequence problem

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The degree sequence problem

Problem: Given an integer sequence $\mathbf{d} = (d_1, \ldots, d_n)$ determine if there exists a graph G with $\mathbf{d}$ as its sequence of degrees.

If such a G exists then $\mathbf{d}$ is said to be *graphic*, and G is called a *realization*.

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Havel (1955) and Hakimi (1962) gave an algorithm to decide.

$(3,3,3,3,3,3) \rightarrow (2,2,2,3,3) = (3,3,2,2,2) \rightarrow (2,1,1,2) = (2,2,1,1) \rightarrow (1,0,1) = (1,1,0) \rightarrow (0,0)$
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To construct a realization, work backwards using simple edge augmentations.

Erdős-Gallai criterion

Theorem

[Erdős, Gallai (1960)]
A nonincreasing sequence of nonnegative integers $\mathbf{d} = (d_1, \ldots, d_n)$ $(n \geq 2)$ *is graphic if, and only if,* $\sum_{i=1}^{n} d_i$ *is even and for each integer* k, $1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min\{k, d_i\}.$$

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The degrees of the first k vertices are "absorbed" within k-subset and the degrees of remaining vertices. A necessary condition which is also sufficient!

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A new constructive (and short) proof of sufficiency given by Tripathi, Venugopalan and West — learn more next month at Albertson Conference at Smith College from West's lecture.

Theorem (Erdős, Gallai)

For a graphic $\mathbf{d}$, $\sum_{i=1}^{n} d_i \geq 2(n-1)$ *if and only if there exists a connected* G *realizing* $\mathbf{d}$.

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(Necessity) Pick the realization of $\mathbf{d}$ with the fewest number of components. If this number is 1, then we are done. Otherwise one of the components contains a cycle. Performing a simple edge-exchange allows us to move to a realization with fewer components.$\square$

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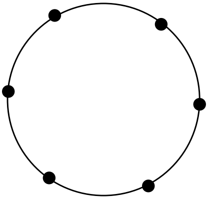
$(2,2,2,2,2,2)$ is potentially K^3-graphic.



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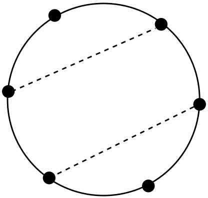

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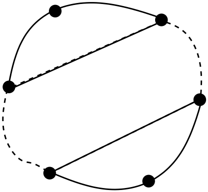


Problem

Given a subgraph F, determine the least even integer m s.t. $\Sigma d_i \geq m \Rightarrow \mathbf{d}$ is potentially F-graphic.

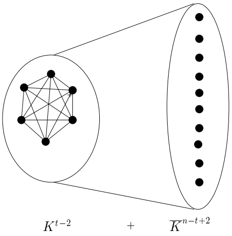
Denote m by $\sigma(F, n)$.

Erdős, Jacobson, Lehel Conjecture

Conjecture

(EJL - 1991) For n sufficiently large,
$\sigma(K^t, n) = (t-2)(2n-t+1)+2$.

Lower bound arises from considering:



$\mathbf{d} = ((n-1)^{t-2}, (t-2)^{n-t+2})$

Erdős, Jacobson, Lehel Conjecture

Conjecture settled:

- $t = 3$ Erdős, Jacobson, & Lehel(1991),
- $t = 4$ Gould, Jacobson, & Lehel(1999), Li & Song(1998),
- $t = 5$ Li & Song(1998),
- $t \geq 6$ Li, Song, & Luo(1998)
- $t \geq 3$ S.(2005), Ferrara, Gould, S. (2009+) - purely graph-theoretic proof.

Theorem

For n sufficiently large, $\sigma(K^t, n) = (t-2)(2n-t+1)+2$.

Sketch of our proof

Uses induction on t.

Sketch of our proof

Erdős-Gallai guarantees enough vertices of high degree.

Lemma

If $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ *is a graphic sequence such that* $\Sigma d_i \geq (t-2)(2n-t+1)+2$ *and* $n \geq t$*, then* $d_t \geq t-1$.

PROOF: By way of contradiction and applying the Erdős-Gallai criteria:

$$
\begin{aligned}
\sum_{i=1}^{n} d_i &= \sum_{i=1}^{t-1} d_i + \sum_{i=t}^{n} d_i \\
&\overbrace{\leq}^{E-G} \left((t-1)(t-2) + \sum_{i=t}^{n} min\{t-1, d_i\}\right) + \sum_{i=t}^{n} d_i \\
&= t^2 - 3t + 2 + 2\sum_{i=t}^{n} d_i \\
&\leq t^2 - 3t + 2 + 2(n-t+1)(t-2) \\
&= (t-2)(2n-t+1).
\end{aligned}
$$

For all $n \geq t$, this contradicts the given degree sum and the result follows.□

Sketch of our proof

Uses notion of an edge-exhange.
Edge-exchange allows us to place desired subgraph on vertices of highest degree and “build” K^t from smaller clique guaranteed by inductive hypothesis.

Extending the EJL-conjecture to an arbitrary graph F

Let F be a forbidden subgraph.
Let $\alpha(F)$ denote the independence number of F and define:

$$u := u(F) = |V(F)| - \alpha(F) - 1,$$

and

$$s := s(F) = min\{\Delta(H) : H \subset F, |H| = \alpha(F) + 1\}.$$

Consider the following sequence,

$$\pi(F, n) = ((n-1)^u, (u + s - 1)^{n-u}).$$

An example

Consider $F = K_{6,6}$. Then,

$$u(K_{6,6}) = |V(K_{6,6})| - \alpha(K_{6,6}) - 1 = 12 - 6 - 1 = 5$$

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Thus,

$$\pi(K_{6,6}, n) = ((n-1)^5, (5+4-1)^{n-5}) = ((n-1)^5, 8^{n-5}).$$

A General Lower Bound

If F' is a subgraph of F then $\sigma(F', n) \leq \sigma(F, n)$ for every n.

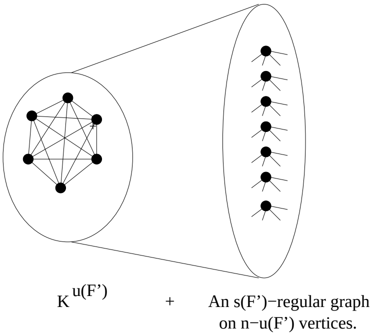
Proposition (Ferrara, S. - '09)

Given a graph F and n sufficiently large then,

$$\sigma(F, n) \quad \geq \quad max\{n(2u(F') + s(F') - 1) | F' \subseteq F\} \tag{1}$$

Proof of Lower Bound

PROOF: Let $F' \subseteq F$ be the subgraph which achieves the max. Consider,



$u(F') = |V(F')| - \alpha(F') - 1$
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A Stronger Lower Bound

Let $v_i(H)$ be the number of vertices of degree i in H. Let $M_i(H)$ denote the set of induced subgraphs on $\alpha+1$ vertices with $v_i(H) > 0$.

For all $i,\ s \leq i \leq \alpha - 1$ define:

$m_i = min_{M_i(H)}\{v_i(H) : H \in M_i(H)\}$

$n_s = m_s - 1$ and $n_i = min\{m_i - 1, n_{i-1}\}$

Finally, set $\delta_{\alpha-1} = n_{\alpha-1}$ and for all $i,\ s \leq i \leq \alpha - 2$ define $\delta_i = n_i - n_{i+1}$ and

$$\begin{aligned} \pi^*(F,n) &= ((n-1)^u, (u+\alpha-1)^{\delta_{\alpha-1}}, (u+\alpha-2)^{\delta_{\alpha-2}}, \ldots \\ &\quad (u+s)^{\delta_s}, (u+s-1)^{n-u-\Sigma\delta_i}). \end{aligned}$$

An Example

Let $F = K_{6,6}$.

Then $u(K_{6,6}) = 12 - 6 - 1 = 5$ and $s(K_{6,6}) = 4$.

$m_4 = 3$ and $m_5 = 2$

$n_4 = m_4 - 1 = 2$ and $n_5 = min\{m_5 - 1, n_4\} = 1$

$\delta_5 = n_5 = 1$ and $\delta_4 = n_5 - n_4 = 1$

Thus,

$$\pi^*(K_{6,6}, n) = ((n-1)^5, 10, 9, 8^{n-7})$$

A Stronger Lower Bound

Theorem (Ferrara, S. - '09)

Given a graph F and n sufficiently large then,

$$\sigma(F,n) \geq max\{sum\ of\ \ terms\ of\ \pi^*(F',n) + 2|F' \subseteq F\}$$

When Does Equality Hold?

- **cliques**
- **complete bipartite graphs** Chen, Li, Yin '04; Gould, Jacobson, Lehel '99; Li, Yin '02
- **complete multipartite graphs** Chen, Yin '08; G. Chen, Ferrara, Gould, S. '08; Ferrara, Gould, S. '08
- **matchings** Gould, Jacobson, Lehel '99
- **cycles** Lai '04
- **(generalized) friendship graph** Ferrara, Gould, S. '06, (Chen, S., Yin '08)
- **clique minus an edge** Lai '01; Li, Mao, Yin '05
- **disjoint union of cliques** Ferrara '08

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(weaker version) Given a graph F, let $\epsilon > 0$. Then there exists an $n_0 = n_0(\epsilon, F)$ such that for any $n > n_0$

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Conjecture (strong form) holds for **graphs with independence number 2** (Ferrara, S. - '09)

WAKE UP!

An example of a generalized degree sequence problem

Is the following graphic?

$$< \mathbb{V}, \mathbf{d}, D > = < \{V_1, V_2\}, (5^4, 3^8), \begin{bmatrix} 6 & 8 \\ 8 & 8 \end{bmatrix} >$$

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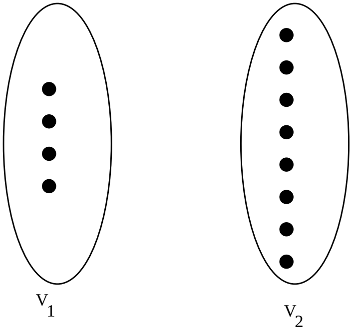

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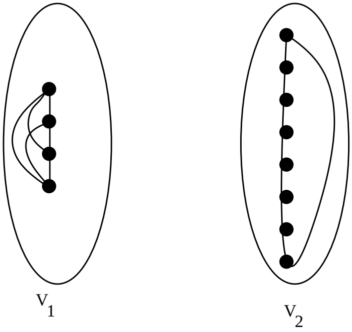

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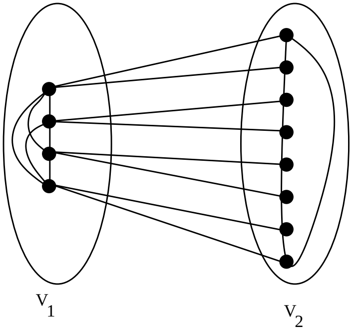

Let $\mathbf{d} = (d_1^{v_1}, d_2^{v_2}, \ldots, d_k^{v_k})$ where $v_i = |V_i|$ and so V_i is the set of vertices of degree d_i. Let $\mathbb{V} = \{V_1, \ldots, V_k\}$. Let $D = (d_{ij})$ be a $k \times k$ matrix, with d_{ij} denoting the number of edges between V_i and V_j; d_{ii} is the number of edges contained entirely within V_i.

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Joint degree-matrix graphic realization problem
Given $< \mathbb{V}, \mathbf{d}, D >$, decide whether a simple graph G exists such that, for all i, each vertex in V_i has degree d_i, and, for $i \neq j$, there are exactly d_{ij} edges between V_i and V_j, while, for all i, there are exactly d_{ii} edges contained in V_i.

Amanatidis, Green and Mihail (AGM) have shown that the following natural necessary conditions for a realization to exist are also sufficient. The conditions are:

Degree feasibility: $2d_{ii} + \Sigma_{j \in [k], j \neq i} d_{ij} = v_i d_i$, for all $1 \leq i \leq k$, and

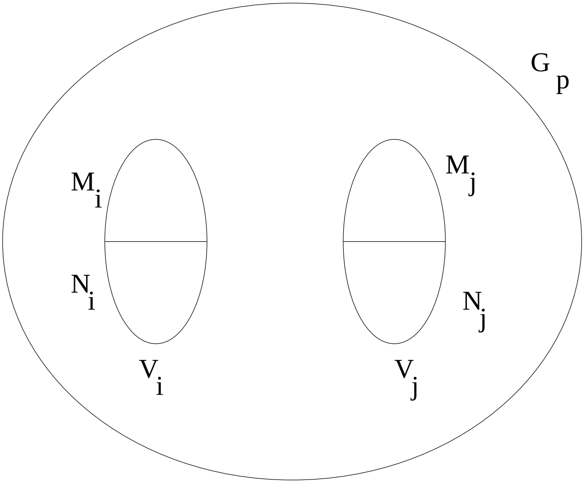
Matrix feasibility: D is a symmetric matrix with non-negative integral entries, $d_{ij} \leq v_i v_j$, for all $1 \leq i \leq k$, and $d_{ii} \leq \binom{v_i}{2}$, for all $1 \leq i \leq k$.

AGM's algorithmic proof

Algorithm rests on a balanced degree invariant. It starts with the empty graph and adds one edge at a time while keeping the difference between any two vertex degrees in a given V_i to at most 1.

While there exists some i, j such that d_{ij} is not satisfied the algorithm adds an edge between V_i and V_j.

AGM algorithm



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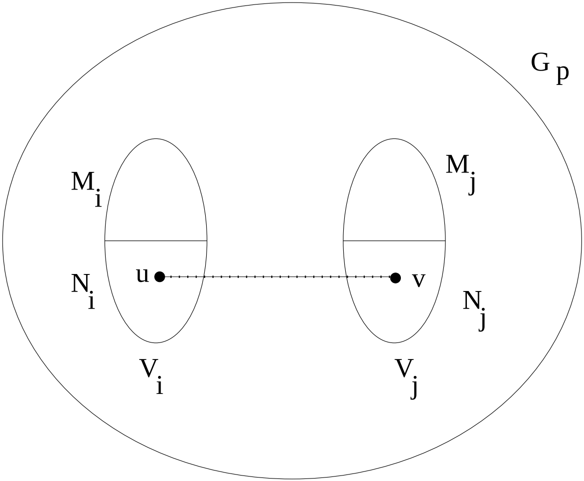

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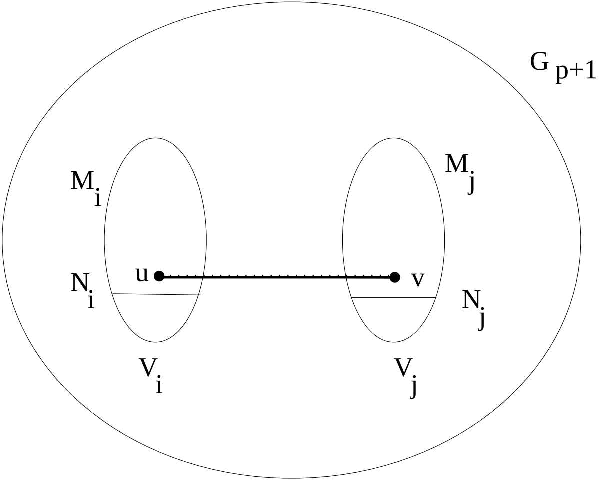

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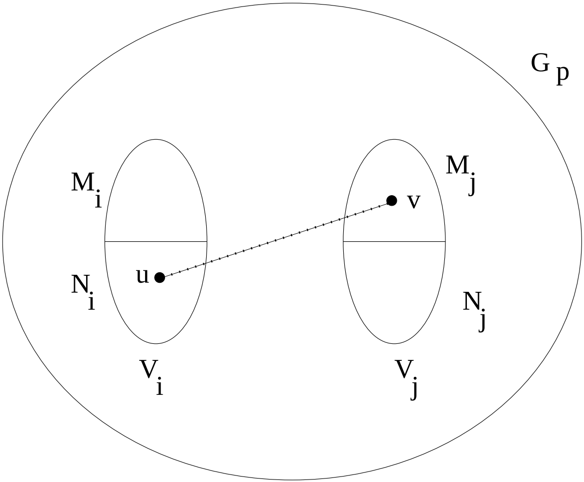

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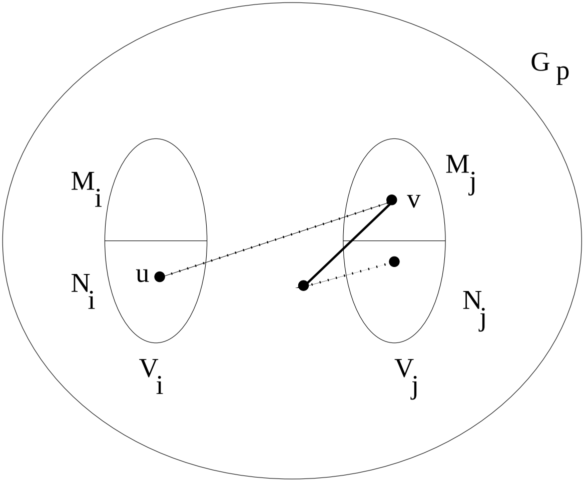

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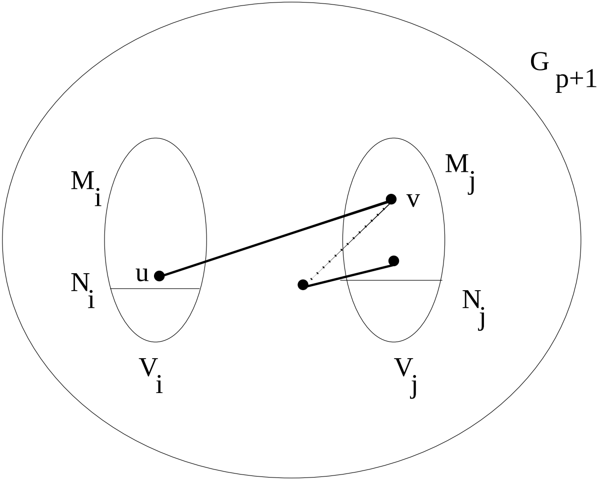

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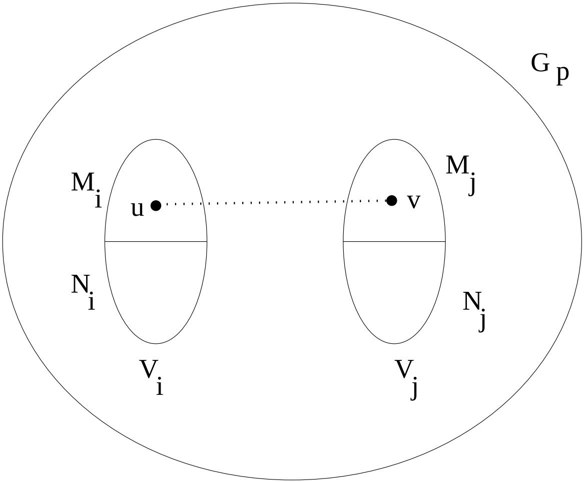

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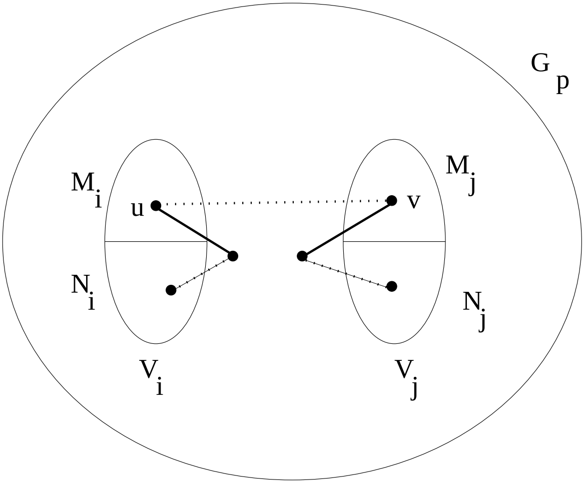

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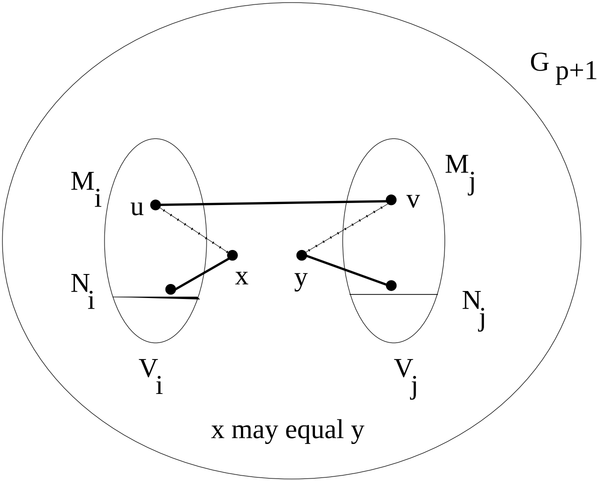

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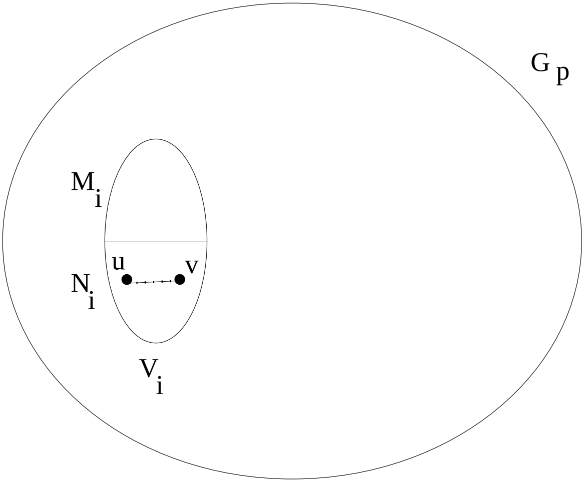

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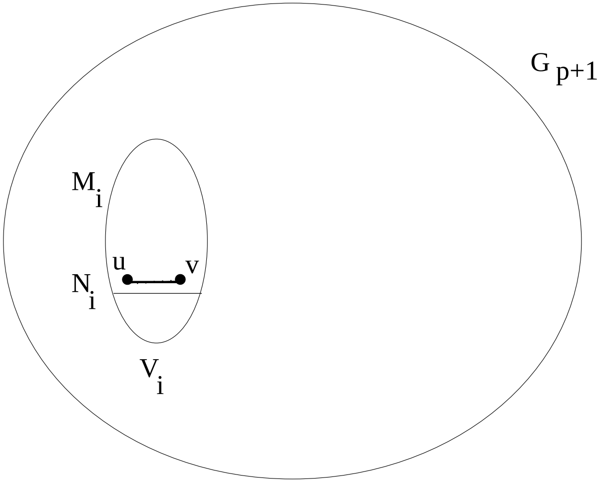

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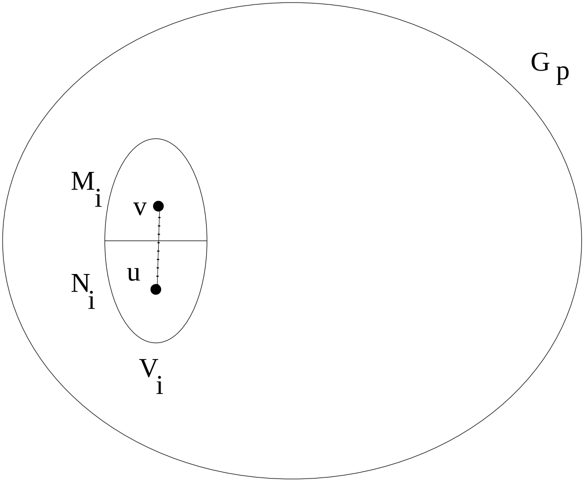

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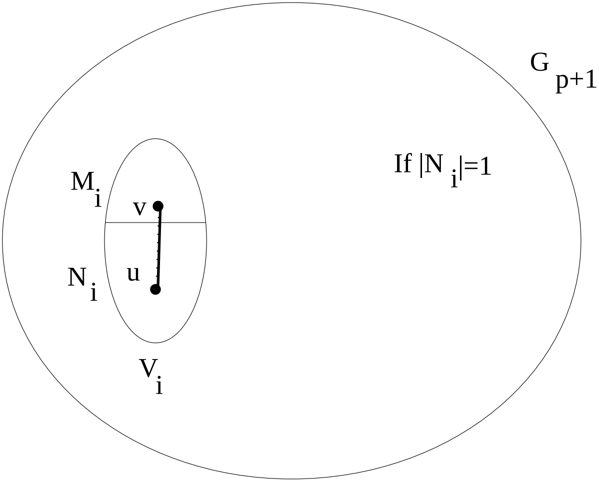

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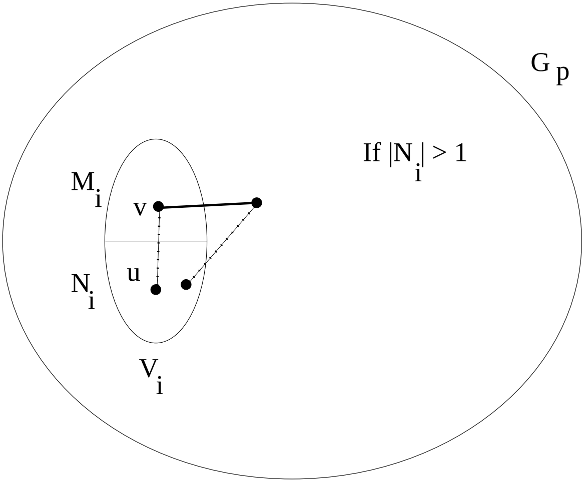

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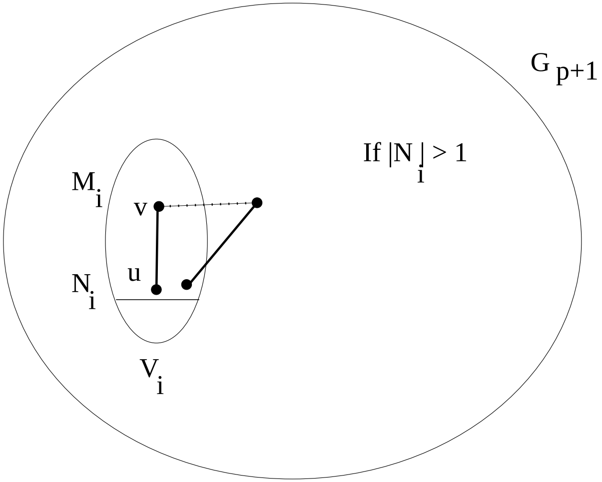

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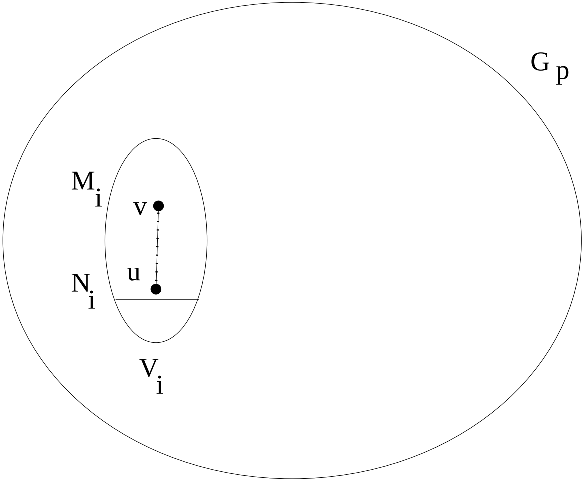

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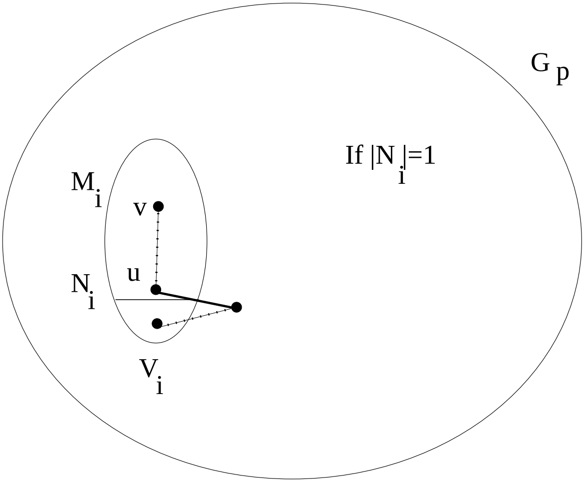

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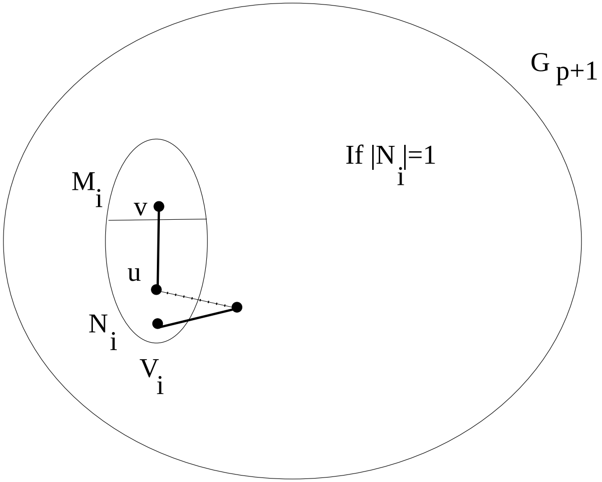

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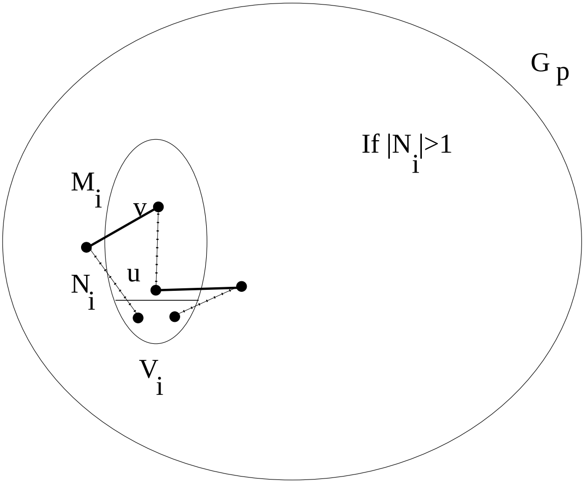

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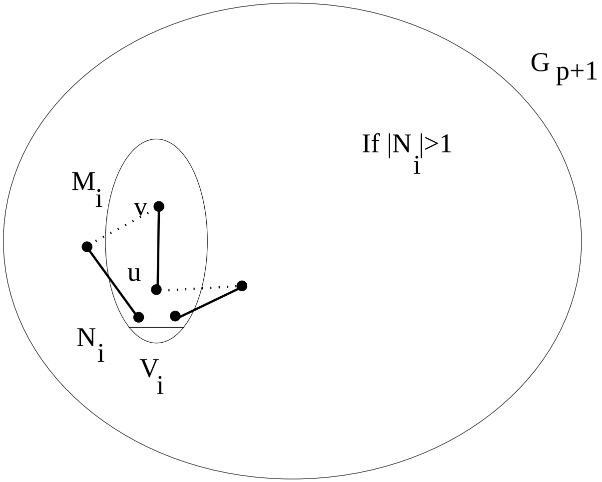

Theorem (Joint Degree-Matrix Realization Theorem - AGM)

Given $< \mathbb{V}, \mathbf{d}, D >$, *if degree and matrix feasibility hold, then a graph* G *exists that realizes* $< \mathbb{V}, \mathbf{d}, D >$. *Furthermore, such a graph can be constructed in time polynomial in* n.

Can one prove our conjectures using the Joint Degree-Matrix Theorem?

Can we prove our conjectures using the AGM-result?

Is $\mathbf{d} = (10^6, 4^8)$ potentially K^6-graphic?

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The value of d_{11} forces, by degree feasibility, $d_{12} = d_{21} = 30$. In turn, by degree feasibility again, we get $d_{22} = 1$. It is now easy to check that matrix feasibility holds. Thus, $\mathbf{d}$ has a realization containing a copy of K^6.

Summary

- Proved the EJL-conjecture
- Generalized the EJL-conjecture and proved a specific case
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Thank you!