# Generalizing the degree sequence problem

John Schmitt

Middlebury College

March 2009
Arizona State University Discrete Mathematics Seminar

# The degree sequence problem

**Problem:** Given an integer sequence $\mathbf{d} = (d_1, \ldots, d_n)$ determine if there exists a graph $G$ with $\mathbf{d}$ as its sequence of degrees.

If such a $G$ exists then $\mathbf{d}$ is said to be *graphic*, and $G$ is called a *realization*.

# An example

Is $\mathbf{d} = (3, 3, 3, 3, 3, 3)$ graphic?

# An example

Is $\mathbf{d} = (3,3,3,3,3,3)$ graphic?

Havel (1955) and Hakimi (1962) gave an algorithm to decide.

$(3,3,3,3,3,3) \rightarrow (2,2,2,3,3) = (3,3,2,2,2) \rightarrow (2,1,1,2) = (2,2,1,1) \rightarrow (1,0,1) = (1,1,0) \rightarrow (0,0)$
As $(0,0)$ is graphic, so is the given.

# An example

Is $\mathbf{d} = (3, 3, 3, 3, 3, 3)$ graphic?

Havel (1955) and Hakimi (1962) gave an algorithm to decide.

$(3, 3, 3, 3, 3, 3) \to (2, 2, 2, 3, 3) = (3, 3, 2, 2, 2) \to (2, 1, 1, 2) =$
$(2, 2, 1, 1) \to (1, 0, 1) = (1, 1, 0) \to (0, 0)$
As $(0, 0)$ is graphic, so is the given.

To construct a realization, work backwards using simple edge augmentations.

# Erdős-Gallai criterion

## Theorem

*[Erdős, Gallai (1960)]*
*A nonincreasing sequence of nonnegative integers* $\mathbf{d} = (d_1, \ldots, d_n)$ $(n \geq 2)$ *is graphic if, and only if,* $\sum_{i=1}^{n} d_i$ *is even and for each integer* $k$, $1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min\{k, d_i\}.$$

# Erdős-Gallai criterion

## Theorem

*[Erdős, Gallai (1960)]*
*A nonincreasing sequence of nonnegative integers* $\mathbf{d} = (d_1, \ldots, d_n)$ $(n \geq 2)$ *is graphic if, and only if,* $\sum_{i=1}^{n} d_i$ *is even and for each integer* $k$, $1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min\{k, d_i\}.$$

The degrees of the first $k$ vertices are "absorbed" within $k$-subset and the degrees of remaining vertices. A necessary condition which is also sufficient!

### Theorem (Erdős, Gallai)

*For a graphic* $\mathbf{d}$, $\sum_{i=1}^{n} d_i \geq 2(n-1)$ *if and only if there exists a connected* $G$ *realizing* $\mathbf{d}$.

### Theorem (Erdős, Gallai)

*For a graphic* $\mathbf{d}$, $\sum_{i=1}^{n} d_i \geq 2(n-1)$ *if and only if there exists a connected* $G$ *realizing* $\mathbf{d}$.

Proof: (Sufficiency) If there exists a connected realization then $G$ contains a spanning tree. Thus $G$ has $n-1$ edges and so $\sum_{i=1}^{n} d_i \geq 2(n-1)$.

## Theorem (Erdős, Gallai)

*For a graphic* $\mathbf{d}$, $\sum_{i=1}^{n} d_i \geq 2(n-1)$ *if and only if there exists a connected* $G$ *realizing* $\mathbf{d}$.

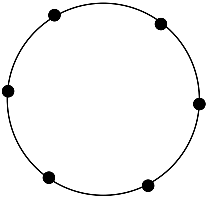
Proof: (Sufficiency) If there exists a connected realization then $G$ contains a spanning tree. Thus $G$ has $n-1$ edges and so $\sum_{i=1}^{n} d_i \geq 2(n-1)$.

(Necessity) Pick the realization of $\mathbf{d}$ with the fewest number of components. If this number is 1, then we are done. Otherwise one of the components contains a cycle. Performing a simple edge-exchange allows us to move to a realization with fewer components.$\square$

For a subgraph $F$, $\mathbf{d}$ is said to be potentially $F$-graphic if there exists a realization of $\mathbf{d}$ containing $F$.

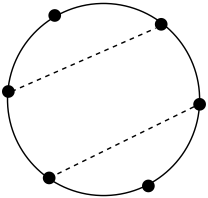
For a subgraph $F$, $\mathbf{d}$ is said to be potentially $F$-graphic if there exists a realization of $\mathbf{d}$ containing $F$.

$(2, 2, 2, 2, 2, 2)$ is potentially $K^3$-graphic.



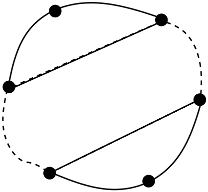
For a subgraph $F$, $\mathbf{d}$ is said to be potentially $F$-graphic if there exists a realization of $\mathbf{d}$ containing $F$.

$(2, 2, 2, 2, 2, 2)$ is potentially $K^3$-graphic.



For a subgraph $F$, $\mathbf{d}$ is said to be potentially $F$-graphic if there exists a realization of $\mathbf{d}$ containing $F$.

$(2, 2, 2, 2, 2, 2)$ is potentially $K^3$-graphic.



## Problem

*Given a subgraph $F$, determine the least even integer $m$ s.t. $\Sigma d_i \geq m \Rightarrow \mathbf{d}$ is potentially $F$-graphic.*

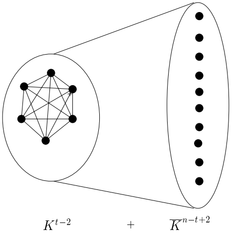
Denote $m$ by $\sigma(F, n)$.

# Erdős, Jacobson, Lehel Conjecture

## Conjecture

*(EJL - 1991) For $n$ sufficiently large,*
$\sigma(K^t, n) = (t-2)(2n-t+1)+2.$

Lower bound arises from considering:




$\mathbf{d} = ((n-1)^{t-2}, (t-2)^{n-t+2})$

# Erdős, Jacobson, Lehel Conjecture

Conjecture settled:

- $t = 3$ Erdős, Jacobson, & Lehel(1991),
- $t = 4$ Gould, Jacobson, & Lehel(1999), Li & Song(1998),
- $t = 5$ Li & Song(1998),
- $t \geq 6$ Li, Song, & Luo(1998)
- $t \geq 3$ S.(2005), Ferrara, Gould, S. (2009+) - purely graph-theoretic proof.

**Theorem**

*For $n$ sufficiently large,* $\sigma(K^t, n) = (t-2)(2n-t+1)+2$.

# Sketch of our proof

- Uses induction on $t$.
- Erdős-Gallai guarantees enough vertices of high degree.
- Uses notion of an edge-exhange.
- Edge-exchange allows us to place desired subgraph on vertices of highest degree and "build" $K^t$ from smaller clique guaranteed by inductive hypothesis.

## Extending the EJL-conjecture to an arbitrary graph $F$

Let $F$ be a forbidden subgraph.
Let $\alpha(F)$ denote the independence number of $F$ and define:

$$u := u(F) = |V(F)| - \alpha(F) - 1,$$

and

$$s := s(F) = min\{\Delta(H) : H \subset F, |H| = \alpha(F) + 1\}.$$

Consider the following sequence,

$$\pi(F, n) = ((n-1)^u, (u+s-1)^{n-u}).$$

# A General Lower Bound

If $F'$ is a subgraph of $F$ then $\sigma(F', n) \leq \sigma(F, n)$ for every $n$. Let $\sigma(\pi)$ denote the sum of the terms of $\pi$.

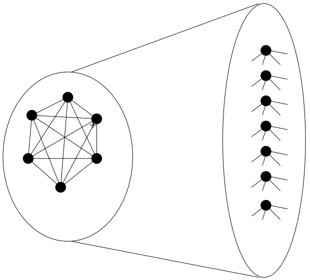
**Proposition (Ferrara, S. - 09)**

*Given a graph $F$ and $n$ sufficiently large then,*

$$\begin{aligned} \sigma(F,n) &\geq max\{\sigma(\pi(F',n)) + 2 | F' \subseteq F\} & (1) \\ &= max\{n(2u(F') + s(F') - 1) | F' \subseteq F\} & (2) \end{aligned}$$

# Proof of Lower Bound

PROOF: Let $F' \subseteq F$ be the subgraph which achieves the max. Consider,




$$u(F') = |V(F')| - \alpha(F') - 1$$
$$s(F') = min\{\Delta(H) : H \subset F', |H| = \alpha(F') + 1\}$$

# A Stronger Lower Bound

Let $v_i(H)$ be the number of vertices of degree $i$ in $H$. Let $M_i(H)$ denote the set of induced subgraphs on $\alpha+1$ vertices with $v_i(H)>0$.

For all $i,\ s \leq i \leq \alpha-1$ define:

$m_i = min_{M_i(H)}\{\text{vertices of degree at least i}\}$

$n_s = m_s - 1$ and $n_i = min\{m_i - 1, n_{i-1}\}$

Finally, set $\delta_{\alpha-1} = n_{\alpha-1}$ and for all $i,\ s \leq i \leq \alpha-2$ define $\delta_i = n_i - n_{i+1}$ and

$$\begin{aligned}\pi^*(F,n) &= ((n-1)^u, (u+\alpha-1)^{\delta_{\alpha-1}}, (u+\alpha-2)^{\delta_{\alpha-2}}, \ldots \\ &\quad (u+s)^{\delta_s}, (u+s-1)^{n-u-\Sigma\delta_i}).\end{aligned}$$

# An Example

Let $F = K_{6,6}$.

Then $u(K_{6,6}) = 12 - 6 - 1 = 5$ and $s(K_{6,6}) = 4$.

$m_4 = 3$ and $m_5 = 2$

$n_4 = m_4 - 1 = 2$ and $n_5 = min\{m_5 - 1, n_4\} = 1$

$\delta_5 = n_5 = 1$ and $\delta_4 = n_5 - n_4 = 1$

Thus,

$$\pi^*(K_{6,6}, n) = ((n-1)^5, 10, 9, 8^{n-7})$$

# A Stronger Lower Bound

**Theorem (Ferrara, S. - 09)**

*Given a graph F and n sufficiently large then,*

$$\sigma(F,n) \geq max\{\sigma(\pi^*(F',n)) + 2 | F' \subseteq F\}$$

# When Does Equality Hold?

- **cliques**
- **complete bipartite graphs** Chen, Li, Yin '04; Gould, Jacobson, Lehel '99; Li, Yin '02
- **complete multipartite graphs** Chen, Yin '08; G. Chen, Ferrara, Gould, S. '08; Ferrara, Gould, S. '08
- **matchings** Gould, Jacobson, Lehel '99
- **cycles** Lai '04
- **(generalized) friendship graph** Ferrara, Gould, S. '06, (Chen, S., Yin '08)
- **clique minus an edge** Lai '01; Li, Mao, Yin '05
- **disjoint union of cliques** Ferrara '08

# Our Conjecture

**Conjecture**

*Given a graph F and n sufficiently large then,*

$$\sigma(F, n) = max\{\sigma(\pi^*(F', n)) + 2 | F' \subseteq F\}$$

**Conjecture**

*(weaker version) Given a graph F, let $\epsilon > 0$. Then there exists an $n_0 = n_0(\epsilon, F)$ such that for any $n > n_0$*

$$\sigma(F, n) \leq max\{(n(2u(F') + d(F') - 1 + \epsilon) | F' \subseteq F\}.$$

# Our Conjecture

## Conjecture

*Given a graph F and n sufficiently large then,*

$$\sigma(F,n) = max\{\sigma(\pi^*(F',n)) + 2 | F' \subseteq F\}$$

## Conjecture

*(weaker version) Given a graph F, let* $\epsilon > 0$. *Then there exists an* $n_0 = n_0(\epsilon, F)$ *such that for any* $n > n_0$

$$\sigma(F,n) \leq max\{(n(2u(F') + d(F') - 1 + \epsilon) | F' \subseteq F\}.$$

Conjecture (strong form) holds for **graphs with independence number 2** (Ferrara, S. - '09)

## An example of the generalized problem

Is the following graphic?

$$< \mathbb{V}, \mathbf{d}, D > = < \{V_1, V_2\}, (5^4, 3^8), \begin{bmatrix} 6 & 8 \\ 8 & 8 \end{bmatrix} >$$

# An example of the generalized problem

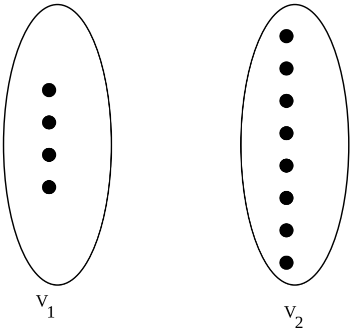
Is the following graphic?

$$< \mathbb{V}, \mathbf{d}, D > = < \{V_1, V_2\}, (5^4, 3^8), \begin{bmatrix} 6 & 8 \\ 8 & 8 \end{bmatrix} >$$

# An example of the generalized problem

Is the following graphic?

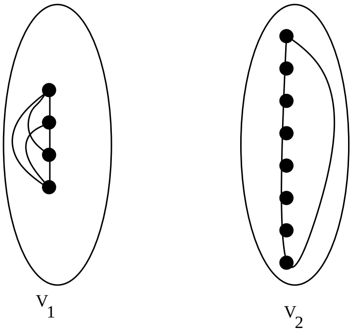
$$< \mathbb{V}, \mathbf{d}, D > = < \{V_1, V_2\}, (5^4, 3^8), \begin{bmatrix} 6 & 8 \\ 8 & 8 \end{bmatrix} >$$



# An example of the generalized problem

Is the following graphic?

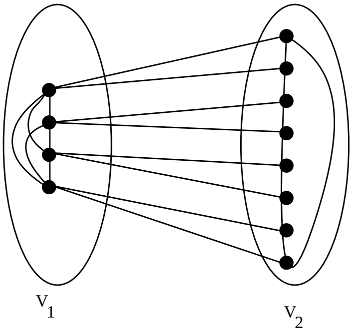
$$< \mathbb{V}, \mathbf{d}, D > = < \{V_1, V_2\}, (5^4, 3^8), \begin{bmatrix} 6 & 8 \\ 8 & 8 \end{bmatrix} >$$



# An example of the generalized problem

Is the following graphic?

$$< \mathbb{V}, \mathbf{d}, D > = < \{V_1, V_2\}, (5^4, 3^8), \begin{bmatrix} 6 & 8 \\ 8 & 8 \end{bmatrix} >$$



Let $\mathbf{d} = (d_1^{v_1}, d_2^{v_2}, \ldots, d_k^{v_k})$ where $v_i = |V_i|$ and so $V_i$ is the set of vertices of degree $d_i$. Let $\mathbb{V} = \{V_1, \ldots, V_k\}$. Let $D = (d_{ij})$ be a $k \times k$ matrix, with $d_{ij}$ denoting the number of edges between $V_i$ and $V_j$; $d_{ii}$ is the number of edges contained entirely within $V_i$.

Let $\mathbf{d} = (d_1^{v_1}, d_2^{v_2}, \ldots, d_k^{v_k})$ where $v_i = |V_i|$ and so $V_i$ is the set of vertices of degree $d_i$. Let $\mathbb{V} = \{V_1, \ldots, V_k\}$. Let $D = (d_{ij})$ be a $k \times k$ matrix, with $d_{ij}$ denoting the number of edges between $V_i$ and $V_j$; $d_{ii}$ is the number of edges contained entirely within $V_i$.

**Joint degree-matrix graphic realization problem**
Given $< \mathbb{V}, \mathbf{d}, D >$, decide whether a simple graph $G$ exists such that, for all $i$, each vertex in $V_i$ has degree $d_i$, and, for $i \neq j$, there are exactly $d_{ij}$ edges between $V_i$ and $V_j$, while, for all $i$, there are exactly $d_{ii}$ edges contained in $V_i$.

Amanatidis, Green and Mihail (AGM) have shown that the following natural necessary conditions for a realization to exist are also sufficient. The conditions are:

*Degree feasibility:* $2d_{ii} + \Sigma_{j \in [k], j \neq i} d_{ij} = v_i d_i$, for all $1 \leq i \leq k$, and

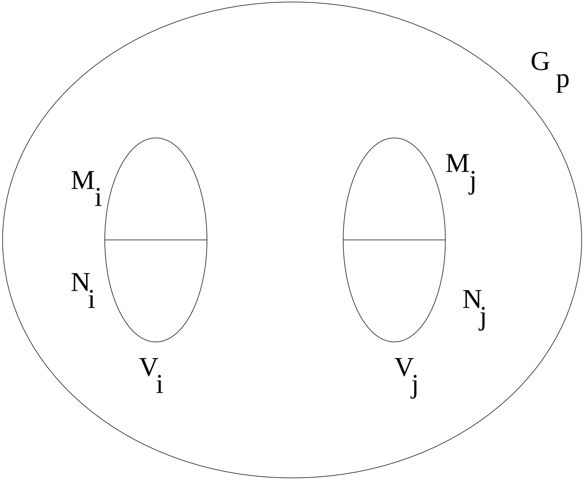
*Matrix feasibility:* $D$ is a symmetric matrix with non-negative integral entries, $d_{ij} \leq v_i v_j$, for all $1 \leq i \leq k$, and $d_{ii} \leq \binom{v_i}{2}$, for all $1 \leq i \leq k$.

# AGM's algorithmic proof

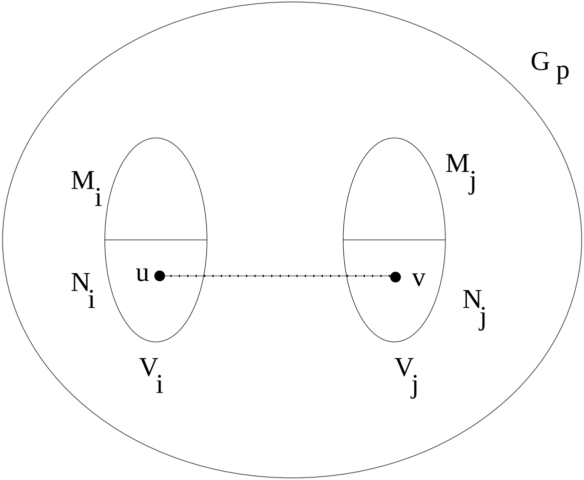
Algorithm rests on a balanced degree invariant. It starts with the empty graph and adds one edge at a time while keeping the difference between any two vertex degrees in a given $V_i$ to at most 1.

While there exists some $i, j$ such that $d_{ij}$ is not satisfied the algorithm adds an edge between $V_i$ and $V_j$.

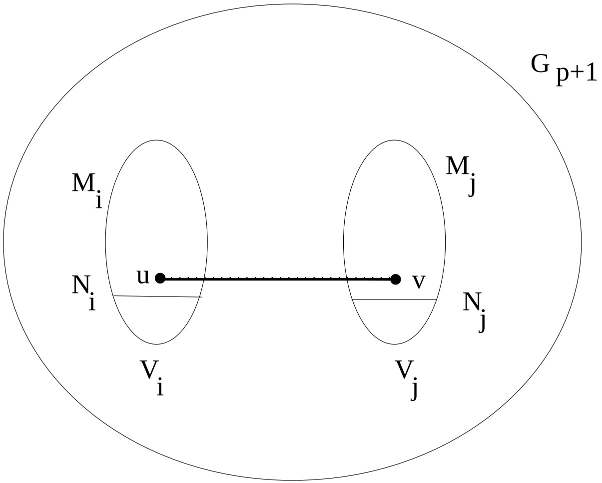
# AGM algorithm



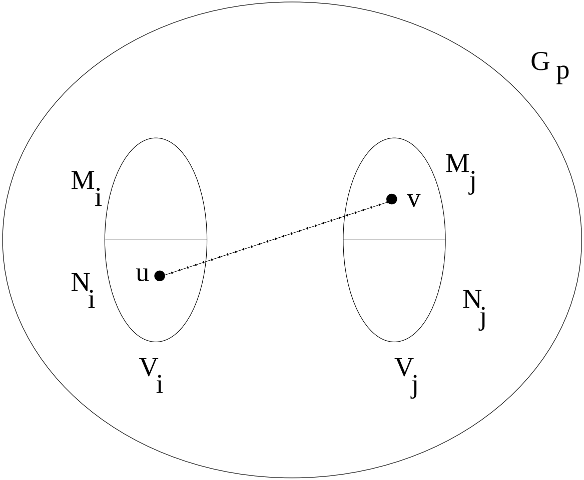
# AGM algorithm



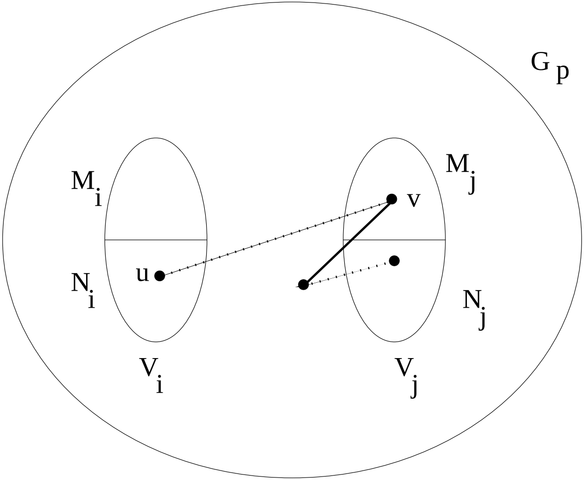
# AGM algorithm



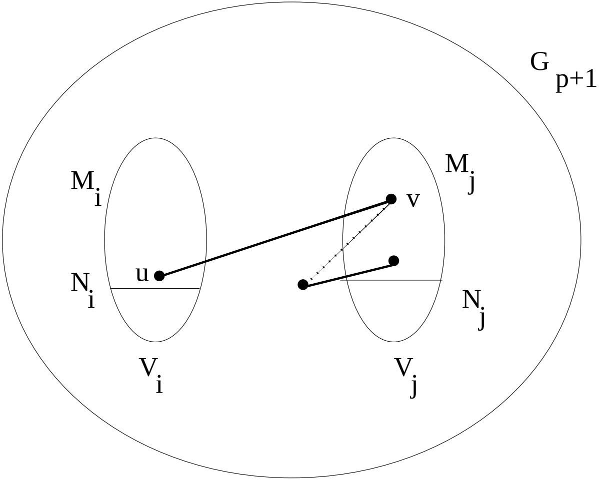
# AGM algorithm



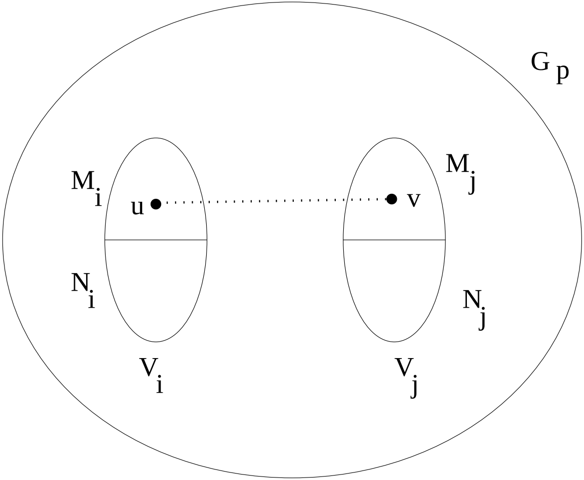
# AGM algorithm



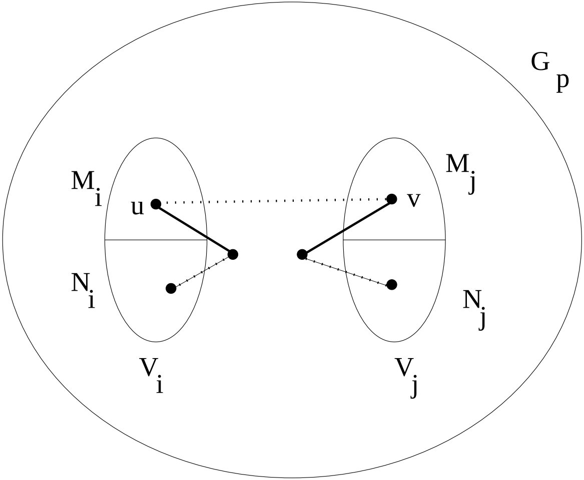
# AGM algorithm



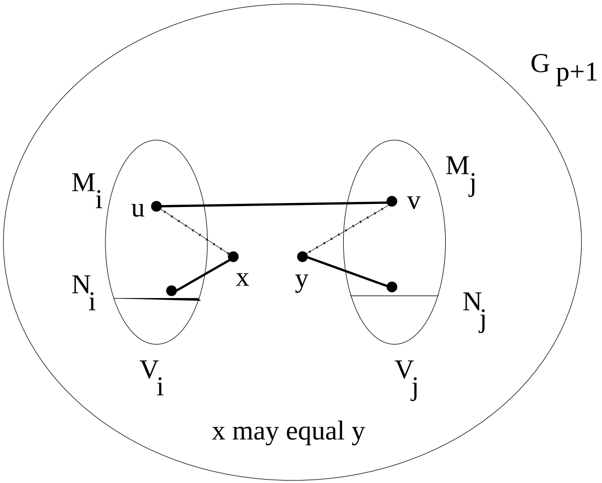
# AGM algorithm



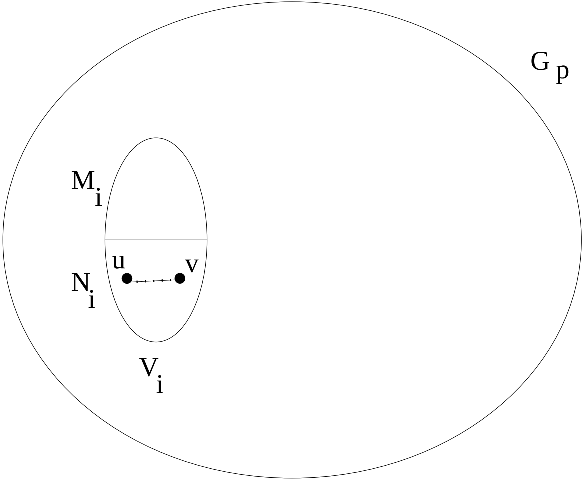
# AGM algorithm



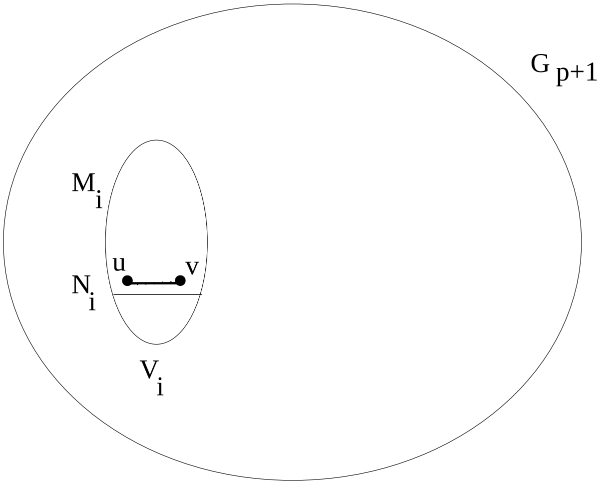
# AGM algorithm



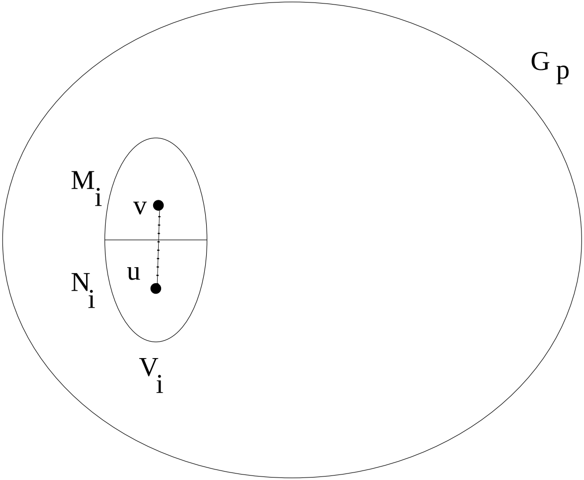
# AGM algorithm



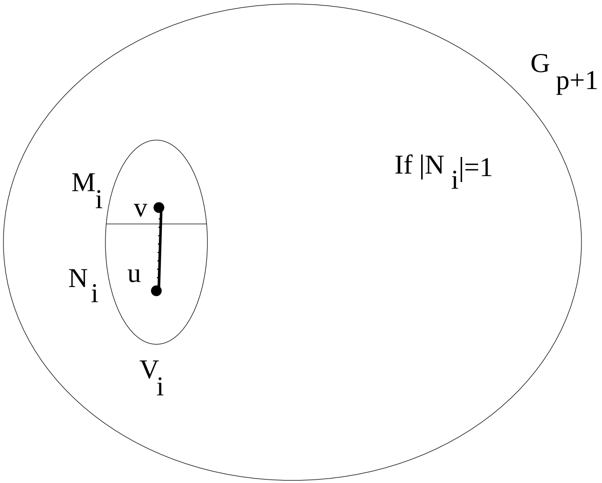
# AGM algorithm



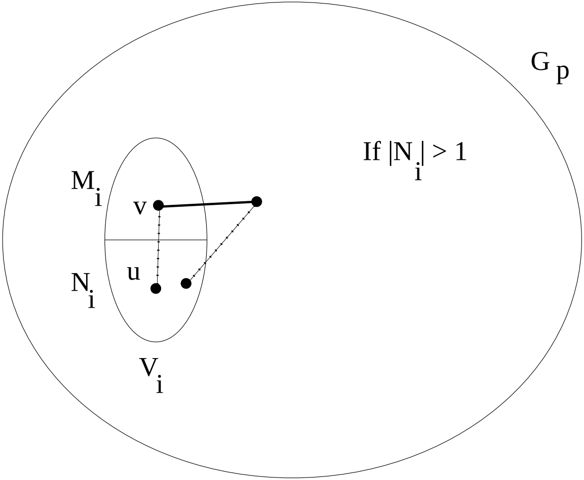
# AGM algorithm



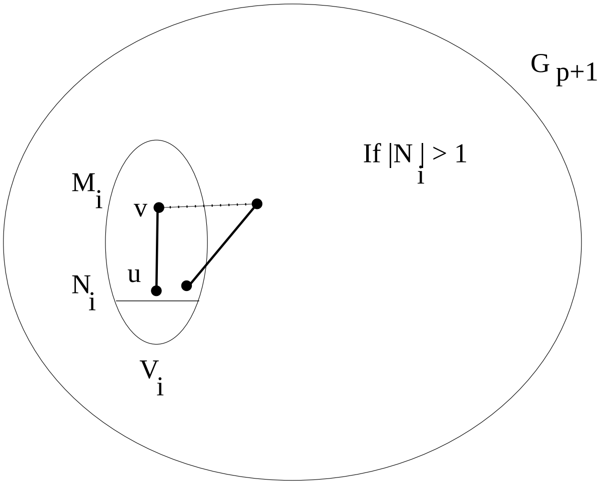
# AGM algorithm



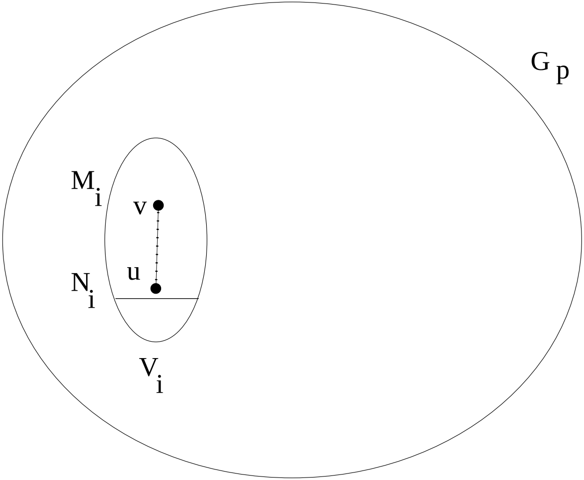
# AGM algorithm



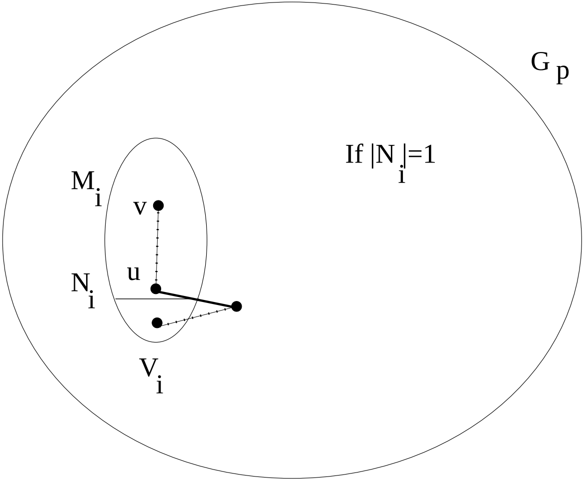
# AGM algorithm



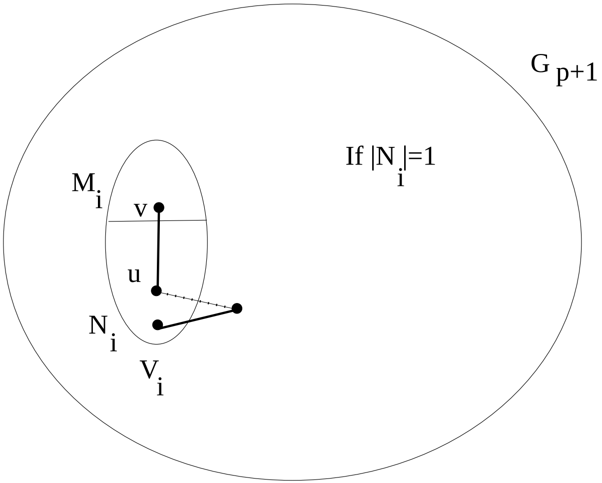
# AGM algorithm



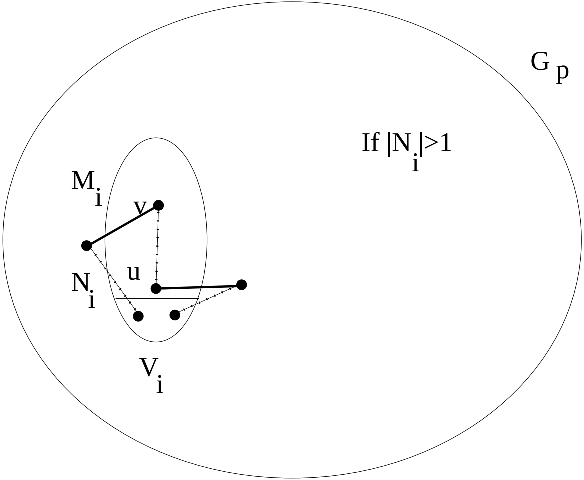
# AGM algorithm



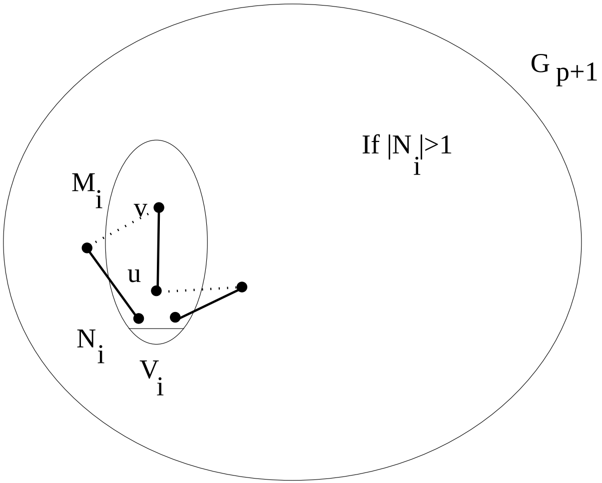
# AGM algorithm



# AGM algorithm



# AGM algorithm



**Theorem (Joint Degree-Matrix Realization Theorem - AGM)**

*Given $< \mathbb{V}, \mathbf{d}, D >$, if degree and matrix feasibility hold, then a graph $G$ exists that realizes $< \mathbb{V}, \mathbf{d}, D >$. Furthermore, such a graph can be constructed in time polynomial in $n$.*

Can one prove our conjectures using the Joint Degree-Matrix Theorem?

## Can we prove our conjectures using the AGM-result?

Is $\mathbf{d} = (10^6, 4^8)$ potentially $K^6$-graphic?

## Can we prove our conjectures using the AGM-result?

Is $\mathbf{d} = (10^6, 4^8)$ potentially $K^6$-graphic? If a realization of $\mathbf{d}$ exists that contains a copy of $K_6$, then, it is known, a realization exists in which the copy of $K^6$ lies on the vertices of highest degree.

# Can we prove our conjectures using the AGM-result?

Is $\mathbf{d} = (10^6, 4^8)$ potentially $K^6$-graphic? If a realization of $\mathbf{d}$ exists that contains a copy of $K_6$, then, it is known, a realization exists in which the copy of $K^6$ lies on the vertices of highest degree. So, by Theorem of AGM, we must simply construct a $D$ for which degree and matrix feasibility hold and for which, in this example, $d_{11} = \binom{6}{2} = 15$.

# Can we prove our conjectures using the AGM-result?

Is $\mathbf{d} = (10^6, 4^8)$ potentially $K^6$-graphic? If a realization of $\mathbf{d}$ exists that contains a copy of $K_6$, then, it is known, a realization exists in which the copy of $K^6$ lies on the vertices of highest degree. So, by Theorem of AGM, we must simply construct a $D$ for which degree and matrix feasibility hold and for which, in this example, $d_{11} = \binom{6}{2} = 15$.
The value of $d_{11}$ forces, by degree feasibility, $d_{12} = d_{21} = 30$. In turn, by degree feasibility again, we get $d_{22} = 1$. It is now easy to check that matrix feasibility holds. Thus, $\mathbf{d}$ has a realization containing a copy of $K^6$.

# Summary

- Proved the EJL-conjecture
- Generalized the EJL-conjecture and proved a specific case
- Joint Degree-Matrix Theorem appears to be a useful tool.
- Can we use it to prove the generalized EJL-conjecture?

# Summary

- Proved the EJL-conjecture
- Generalized the EJL-conjecture and proved a specific case
- Joint Degree-Matrix Theorem appears to be a useful tool.
- Can we use it to prove the generalized EJL-conjecture?

Thank you!