How to beat your friends at the dots-and-boxes game!

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Combinatorial Games

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- are two-player games with alternating play and play continues until player whose turn it is to move has no legal move,
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Examples: Go, Chess, Checkers, Tic-tac-toe, Brussel Sprouts, Clobber, Domineering, Hex, Nim, Snort and DOTS-AND-BOXES.
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**Examples:** Go, Chess, Checkers, Tic-tac-toe, Brussel Sprouts, Clobber, Domineering, Hex, Nim, Snort and **DOTS-AND-BOXES**.

**Not:** Backgammon (has dice), Poker (has card deal), Stratego (has hidden information), Monopoly (has dice and $> 2$ players sometimes)
Dots-and-boxes - how to play

- Played on a rectangular grid with $m$ rows, each containing $n$ dots.
- A move consists of drawing a horizontal or vertical line connecting two dots.
- Upon completing a box, a player ‘claims’ it and moves again.
- Play ends when all possible lines are drawn.
- The winner is the person who has claimed the most boxes. If both players have the same number of boxes, then it is a tie.
Let’s play!

The 9-box game (4 × 4-game).

● ● ● ●
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Let’s play from here (49 moves and turns played) - player A plays odd numbered turns, player B plays even turns:
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What is the **greedy** approach?
Let's play from here (49 moves and turns played) - player A plays odd numbered turns, player B plays even turns:

Don’t take the greedy approach! Let’s make double-dealing moves. Your opponent will be double-crossed (forced to take two boxes with one pen-stroke).
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Chain
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Strategy

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- If you can force your opponent to ‘open’ a long chain, you have control...try to GET CONTROL, man!
- Once you have control, KEEP CONTROL, dude, by (politely) declining 2 boxes of every long chain except the last. (So, you will be last to play.)
To get control

Player A tries to make \# of initial dots + \#double-crossed moves odd.
Player B tries to make \# of initial dots + \#double-crossed moves even.
To get control

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Player B tries to make \# of initial dots + \#double-crossed moves even.
Generally, \# of double-crosses is one less than the \# of long chains.
To get control

Player A tries to make $\#$ of initial dots + $\#$ double-crossed moves odd.
Player B tries to make $\#$ of initial dots + $\#$ double-crossed moves even.
Generally, $\#$ of double-crosses is one less than the $\#$ of long chains.
**LONG CHAIN RULE** Try to make $\#$ of initial dots + $\#$ eventual long chains even if you are A, odd if you are B.
To get control

Player A tries to make \# of initial dots + \# double-crossed moves odd.
Player B tries to make \# of initial dots + \# double-crossed moves even.
Generally, \# of double-crosses is one less than the \# of long chains.

**LONG CHAIN RULE** Try to make \# of initial dots + \# eventual long chains even if you are A, odd if you are B.

**WHY?** \# of initial dots + \# of double-crosses = total \# of turns in game.
Proof of Long Chain Rule

Game board has $m$ rows with $n$ dots each.
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# of dots = $mn$. 
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\# of dots = $mn$.
\# of horizontal moves = $m(n - 1) = mn - m$. 
Proof of Long Chain Rule

Game board has $m$ rows with $n$ dots each.

- # of dots $= mn$.
- # of horizontal moves $= m(n - 1) = mn - m$.
- # of vertical moves $= n(m - 1) = mn - n$. 
Proof of Long Chain Rule

Game board has $m$ rows with $n$ dots each.

# of dots = $mn$.

# of horizontal moves = $m(n - 1) = mn - m$.

# of vertical moves = $n(m - 1) = mn - n$.

# of moves = # vertical + # horizontal = $2mn - m - n$. 
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# of boxes = $(m - 1)(n - 1) = mn - m - n + 1$. 
Proof of Long Chain Rule

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- # moves - # of boxes = $mn - 1$. 

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- \# of vertical moves = $n(m - 1) = mn - n$.
- \# of moves = \# vertical + \# horizontal = $2mn - m - n$.
- \# of boxes = $(m - 1)(n - 1) = mn - m - n + 1$.
- \# moves − \# of boxes = $mn - 1$.
- \# of completed turns = \# of moves − \# of boxes + \# of double-crosses.
Proof of Long Chain Rule

Game board has $m$ rows with $n$ dots each.

# of dots = $mn$.

# of horizontal moves = $m(n - 1) = mn - m$.

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# moves - # of boxes = $mn - 1$.

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# of completed turns = $mn - 1 + # of double-crosses = # of dots - 1 + # of double-crosses$. 
Game board has $m$ rows with $n$ dots each.

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- # of moves = # of vertical + # of horizontal = $2mn - m - n$.
- # of boxes = $(m - 1)(n - 1) = mn - m - n + 1$.
- # of completed turns = # of moves - # of boxes + # of double-crosses.

- # of completed turns = $mn - 1 + $ # of double-crosses = # of dots -1 + # of double-crosses.

Last move of game must complete a box, so final turn is incomplete. Adding this turn to total:
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# of dots = $mn$.
# of horizontal moves = $m(n-1) = mn - m$.
# of vertical moves = $n(m-1) = mn - n$.
# of moves = # vertical + # horizontal = $2mn - m - n$.
# of boxes = $(m-1)(n-1) = mn - m - n + 1$.
# moves -# of boxes = $mn - 1$.
# of completed turns = # of moves -# of boxes +# of double-crosses.
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Last move of game must complete a box, so final turn is incomplete. Adding this turn to total:
# of turns = # of dots +# of double-crosses.
Chain Counting Problems - Puzzle 1

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10 moves made, so it is A’s turn.
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10 moves made, so it is A’s turn. \# dots is 16, an even number. By the long chain rule, A wants an even number of long chains.
The dashed line indicates best move. It ensures 2 chains.
Chain Counting Problems

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Slides available from my homepage:
http://community.middlebury.edu/~jschmitt/.