A dual to the Turán problem

John Schmitt

Middlebury College
joint work with
Ron Gould (Emory University)
Tomasz Łuczak (Adam Mickiewicz University and Emory University)
Oleg Pikhurko (Carnegie Mellon University)

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A dual to the Turán problem
Cycles
Paths, Bipartite Graphs and General Bound
A final problem

Definitions
History

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Definition
A graph \( G \) is \( F \)-saturated if

\[
F \not\subset G \quad \text{and} \quad F \subset G + e \quad \text{for any } e \in E(\overline{G}).
\]
The father of Extremal Graph Theory

Paul Turán

**Turán’s Theorem, 1941** Among the $K_t$-saturated graphs on $n$ vertices, the graph $K_{n_1,n_2,...,n_{t-1}}$, where the $n_i$ are as balanced as possible, has the maximum number of edges.
Theorem Given a graph $F$ with chromatic number $\chi(F)$ at least three, $F$-saturated graphs on $n$ vertices can have at most

$$\left(\frac{\chi(F) - 2}{\chi(F) - 1} + o(1) \right)\left(\begin{array}{c} n \\ 2 \end{array}\right)$$

edges.
Problem

Determine the *minimum number* of edges in an $n$-vertex $F$-saturated graph. We denote this number by $\text{sat}(n,F)$. 
Theorem (Erdős, Hajnal, Moon - 1964)

\[ \text{sat}(n, K_t) = (t - 2)(n - 1) - \binom{t - 2}{2}. \]

Furthermore, the only \( K_t \)-saturated graph with this many edges is \( K_{t-2} + \overline{K}_{n-t+2} \).
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Theorem (Ollmann - ’72, Tuza - ’86)

\[ \text{sat}(n, C_4) = \left\lfloor \frac{3n - 5}{2} \right\rfloor, \quad n \geq 5. \]
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\[ \text{sat}(n, C_4) = \left\lfloor \frac{3n - 5}{2} \right\rfloor, \quad n \geq 5. \]
Theorem (Fisher, Fraughnaugh, Langley - '97)

\[ \text{sat}(n, P_3 - \text{connected}) = \left\lfloor \frac{3n - 5}{2} \right\rfloor. \]

Theorem

\[ \text{sat}(n, C_5) = \left\lceil \frac{10n - 10}{7} \right\rceil, n \neq 21. \]

Fisher, Fraughnaugh, Langley, -'95 gave the upper bound. Y.C. Chen - '09 gave(!) the lower bound.

Problem (FFL)

Determine \( \text{sat}(n, P_4 - \text{connected}) \).
Hamiltonian Cycles

Theorem

$$\text{sat}(n, C_n) = \left\lfloor \frac{3n + 1}{2} \right\rfloor, \; n \geq 53.$$ 

Bondy (’72) showed the lower bound. Clark, Entringer, Crane and Shapiro (’83-’86) gave upper bound based on Isaacs’ flower snarks (girth 5, 6). L. Stacho (’96) gave further constructions based on the Coxeter graph (girth 7).
Hamiltonian Cycles

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Problem (Horák, Širáň -’86)

Is there a maximally non-hamiltonian graph of girth at least 8?
Conjecture (Bollobás - '78)

There exist constants, $c_1, c_2$, such that

$$n + c_1 \frac{n}{l} \leq \text{sat}(n, C_l) \leq n + c_2 \frac{n}{l}.$$ 

Theorem (Barefoot, Clark, Entringer, Porter, Székely, Tuza - '96)

$$(1 + \frac{1}{2l + 8})n \leq \text{sat}(n, C_l)$$
Theorem (Barefoot et al. - ’96)

\[ sat(n, C_l) \leq (1 + \frac{6}{l-3})n + O(l^2) \text{ for } l \text{ odd, } l \geq 9 \]

\[ sat(n, C_l) \leq (1 + \frac{4}{l-2})n + O(l^3) \text{ for } l \text{ even, } l \geq 14 \]
Theorem (Barefoot et al. - '96)

\[ \text{sat}(n, C_l) \leq (1 + \frac{1}{3} \frac{6}{l-3})n + \frac{5l^2}{4} \text{ for } l \text{ odd, } l \geq 9, l \geq 17, n \geq 7l \]

\[ \text{sat}(n, C_l) \leq (1 + \frac{1}{2} \frac{4}{l-2})n + \frac{5l^2}{4} \text{ for } l \text{ even } l \geq 14, l \geq 10, n \geq 3l \]

Theorem

\[ \text{sat}(n, C_l) < \left\lfloor \frac{3n}{2} \right\rfloor + \frac{l^2}{2} \]

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Both small and large
Our Result and Logic of Construction
Łuczak Wheel
Cycle Summary

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A dual to the Turán problem
The Even Łuczak Wheel, $l = 2k + 2 \geq 10$
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The Even Łuczak Wheel, $l = 2k + 2 \geq 10$
Counting Edges of the Łuczak Wheel

For $l = 2k + 2$ and $n \equiv a \mod k$,

$$|E(L - \text{Wheel})| = \left( n - k - a \right) + \frac{n - k - a}{k} + a + \binom{k}{2}.$$

**Theorem**

[GLS] For $k \geq 4$, $l = 2k + 2$, $n \equiv a \mod k$ and $n \geq 3l$,

$$\text{sat}(n, C_l) \leq n \left( 1 + \frac{1}{k} \right) + \frac{k^2 - 3k - 2}{2} - \frac{a}{k}$$

$$\leq n \left( 1 + \frac{2}{l - 2} \right) + \frac{5l^2}{4}.$$
The Odd Łuczak Wheel, $l = 2k + 3 \geq 17$
### $C_l$-saturated graphs of minimum size

<table>
<thead>
<tr>
<th>$l$</th>
<th>$\text{sat}(n, C_l)$</th>
<th>$n \geq$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$n - 1$</td>
<td>3</td>
<td>EHM</td>
</tr>
<tr>
<td>4</td>
<td>$\left\lfloor \frac{3n-5}{2} \right\rfloor$</td>
<td>5</td>
<td>Ollmann; Tuza</td>
</tr>
<tr>
<td>5</td>
<td>$\left\lceil \frac{10n-10}{7} \right\rceil$</td>
<td>21</td>
<td>FFL; Chen</td>
</tr>
<tr>
<td>6</td>
<td>$\leq \frac{3n}{2}$</td>
<td>11</td>
<td>Barefoot et al.</td>
</tr>
<tr>
<td>7</td>
<td>$\leq \frac{7n+12}{5}$</td>
<td>10</td>
<td>Barefoot et al.</td>
</tr>
<tr>
<td>8, 9, 11, 13, 15</td>
<td>$\leq \frac{3n}{2} + \frac{l^2}{2}$</td>
<td>21</td>
<td>GLS</td>
</tr>
<tr>
<td>$\geq 10$ and $\equiv 0 \pmod{2}$</td>
<td>$\leq (1 + \frac{2}{l-2}) n + \frac{5l^2}{4}$</td>
<td>31</td>
<td>Łuczak wheel</td>
</tr>
<tr>
<td>$\geq 17$ and $\equiv 1 \pmod{2}$</td>
<td>$\leq (1 + \frac{2}{l-3}) n + \frac{5l^2}{4}$</td>
<td>71</td>
<td>Łuczak wheel</td>
</tr>
<tr>
<td>$n$</td>
<td>$\left\lfloor \frac{3n+1}{2} \right\rfloor$</td>
<td>20</td>
<td>Bondy; CE, CES</td>
</tr>
</tbody>
</table>
Problem (Barefoot et al. - '96)

Determine the value of $l$ which minimizes $\text{sat}(n, C_l)$ for fixed $n$.

Problem

Are any of these constructions optimal? Can one improve the lower bound?
Other values of \( sat(n, F) \) known for:

- **matchings** (Mader - '73),
- **stars and paths** (Kászonyi and Tuza - '86),
- **hamiltonian path, \( P_n \)** (Frick and Singleton, 05; Dudek, Katona, Wojda - '06)
- **longest path = detour** (Beineke, Dunbar, Frick, '05)
- **disjoint cliques of the same order** (Faudree, Gould, Jacobson, Ferrara - '08)
Difficulties and Hereditary Properties Lacking

Quote from Erdős, Hajnal and Moon:
“One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult.”

- $\text{sat}(n, F) \not\leq \text{sat}(n + 1, F)$
- $\mathcal{F}_1 \subset \mathcal{F}_2 \nRightarrow \text{sat}(n, \mathcal{F}_1) \geq \text{sat}(n, \mathcal{F}_2)$
- $\mathcal{F}' \subset \mathcal{F} \nRightarrow \text{sat}(n, \mathcal{F}') \leq \text{sat}(n, \mathcal{F})$
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- \( \text{sat}(n, F) \not\subseteq \text{sat}(n + 1, F) \)
- \( \mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow \text{sat}(n, \mathcal{F}_1) \geq \text{sat}(n, \mathcal{F}_2) \)
- \( \mathcal{F}' \subset \mathcal{F} \not\Rightarrow \text{sat}(n, \mathcal{F}') \leq \text{sat}(n, \mathcal{F}) \)

- \( \text{sat}(2k - 1, P_4) = k + 1 \) and \( \text{sat}(2k, P_4) = k \)
- \( \text{sat}(n, \{P_5, S_4\}) = n - 1 > \text{sat}(n, P_5) \)
- \( \text{sat}(n, K_4) = 2n - 3 \) but \( \text{sat}(n, K_5 - S_3) \leq \frac{3}{2}n \)
**Theorem (Kászonyi and Tuza - ’86 )**

Let $F$ be a graph. Set

$$u = |V(F)| - \alpha(F) - 1$$

$$s = \min\{e(F') : F' \subseteq F, \alpha(F') = \alpha(F), |V(F')| = \alpha(F) + 1\}.$$

Then

$$\text{sat}(n, F) \leq (u + \frac{s - 1}{2})n - \frac{u(s + u)}{2}.$$

They considered a clique on $u$ vertices joined to an $(s - 1)$-regular graph.
Best Known Lower Bound
Best Known Lower Bound

???

Problem

For an arbitrary graph $F$, determine a non-trivial lower bound on $\text{sat}(n, F)$.
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For an arbitrary graph $F$, determine a non-trivial lower bound on $\text{sat}(n, F)$. 
Let $\text{sat}(n, F, \delta)$ equal minimum number of edges in a graph on $n$ vertices and minimum degree $\delta$ that is $F$-saturated.

**Theorem (Duffus, Hanson - '86)**

\[
\text{sat}(n, K_3, 2) = 2n - 5, \quad n \geq 5.
\]
\[
\text{sat}(n, K_3, 3) = 3n - 15, \quad n \geq 10.
\]

**Problem (Bollobás - '95)**

*Is it true that for every fixed $\delta \geq 1$ one has $\text{sat}(n, K_3, \delta) = \delta n - O(1)$?*
Theorem (Gould, S. - '07)

For integers $t \geq 3$, $n \geq 4t - 4$,

$sat(n, K_{t(2)}) \leq sat(n, K_{t(2)}, 2t - 3) = \left\lceil \frac{(4t - 5)n - 4t^2 + 6t - 1}{2} \right\rceil$.

Problem

Given a fixed graph $F$, for $n$ sufficiently large determine if the function $sat(n, F, \delta)$ is monotonically increasing in $\delta$. 
Theorem (Pikhurko, S. - ’08)

There is a constant $C$ such that for all $n \geq 5$ we have

$$2n - Cn^{3/4} \leq \text{sat}(n, K_{2,3}) \leq 2n - 3.$$
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Proof of Lower Bound

Let $G$ be a $K_{2,3}$-saturated graph.
Proof of Lower Bound

Let $G$ be a $K_{2,3}$-saturated graph. If $\delta(G) \geq 4$, then $|E(G)| \geq 2n$ and we are done.
Let $G$ be a $K_{2,3}$-saturated graph.

If $\delta(G) = 1$ then,
Proof of Lower Bound

If $\delta(G) = 1$ then,

$$|E(G)| \geq 2n - 3.$$
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Proof of Lower Bound

Otherwise, $2 \leq \delta(G) \leq 3$, pick vertex of minimum degree and consider breadth-first search tree.

---

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Otherwise, $2 \leq \delta(G) \leq 3$, pick vertex of minimum degree and consider breadth-first search tree.

Tree has $n - 1$ edges, we must find $n - Cn^{3/4}$ more edges.
Divide and Conquer

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Divide and Conquer
Hit ’em where they’re weakest

$Y_0$ has at most one component which is a tree. Pick up an extra $V_3 - 1$ edges.
More Division and More Conquering

Pick up extra $V_2 - \#(\text{trees in } X_0)$ edges.
More Division and More Conquering
Trees in $X_0$ are connected via a path of length at most three through $V_3$. 

More Hitting Weak Spots
More Hitting Weak Spots

Small degree vertices can only “serve” so many trees of $X_0$. So, sum of large degree vertices is large.

$V_1$

$V_2$

$V_3$

$X_0$

$X_1$

$X_2$

small degree vertices

large degree vertices
Small degree vertices can only “serve” so many trees of $X_0$. So, sum of large degree vertices is large. This allows us to add $\#(\text{trees in } X_0) - O(n^{3/4})$ edges to the count. Completes proof. \hfill $\square$
Recently, Bohman, Fonoberova and Pikhurko have announced the determination of $\text{sat}(n, K_{s_1, s_2, \ldots, s_r})$.

$$\text{sat}(n, K_{s_1, s_2, \ldots, s_r}) = (s_1 + s_2 + \ldots + s_{r-1} - 1 + \frac{s_r - 1}{2} + o(1))n,$$

where $s_1 \leq s_2 \leq \ldots \leq s_r$. 
And Ramsey Numbers

\[ F \to (F_1, \ldots, F_t) \] if any \( t \) coloring of \( E(F) \) contains a monochromatic \( F_i \)-subgraph of color \( i \) for some \( i \in [t] \).

**Conjecture (Hanson and Toft, ’87)**

*Given \( t \geq 2 \) and numbers \( m_i \geq 3, i \in [t] \), let*

\[ \mathcal{F} = \{ F : F \to (K_{m_1}, \ldots, K_{m_t}) \}. \]

*Let \( r = r(K_{m_1}, \ldots, K_{m_t}) \) be the classical Ramsey number. Then*

\[ \text{sat}(n, \mathcal{F}) = (r - 2)(n - 1) - \binom{r - 2}{2}. \]
Many Thanks!!

Talk and results are available online at:
http://community.middlebury.edu/~jschmitt/