A dual to the Turán problem

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joint work with
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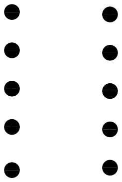
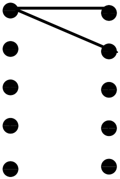
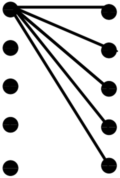
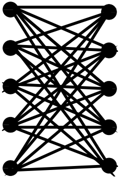
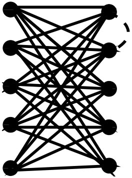
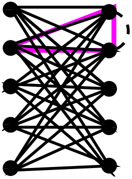


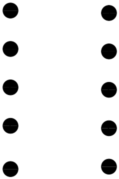


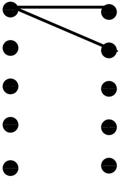


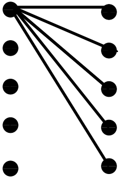


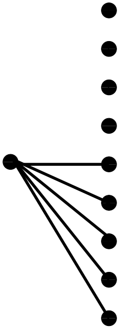


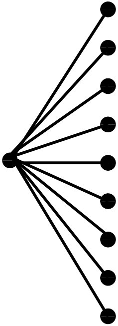


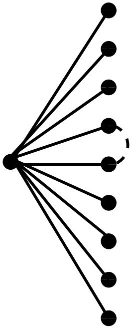


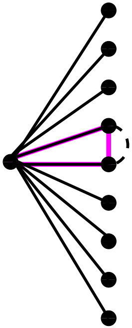












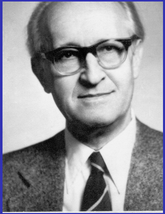
Definition

A graph G is **F-saturated** if

$F \not\subset G$ and

$F \subset G + e$ for any $e \in E(\overline{G})$.

The father of Extremal Graph Theory

Paul Turán 

Turán's Theorem, 1941 Among the K_t-saturated graphs on n vertices, the graph $K_{n_1,n_2,\ldots,n_{t-1}}$, where the n_i are as balanced as possible, has the maximum number of edges.

Erdős-Stone-Simonovits Theorem

Theorem Given a graph F with chromatic number $\chi(F)$ at least three, F-saturated graphs on n vertices can have at most

$$(\frac{\chi(F)-2}{\chi(F)-1}+o(1))\binom{n}{2}$$

edges.

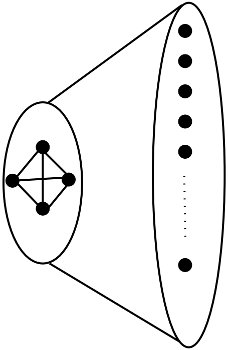
Problem

Determine the minimum number of edges in an n-vertex F-saturated graph. We denote this number by $sat(n, F)$.

Theorem (Erdős, Hajnal, Moon - 1964)

$$sat(n, K_t) = (t-2)(n-1) - \binom{t-2}{2}.$$

Furthermore, the only K_t-saturated graph with this many edges is $K_{t-2} + \overline{K}_{n-t+2}$.



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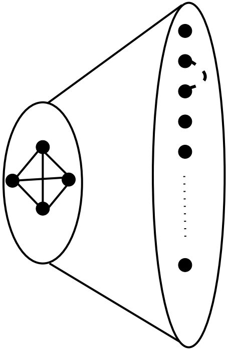

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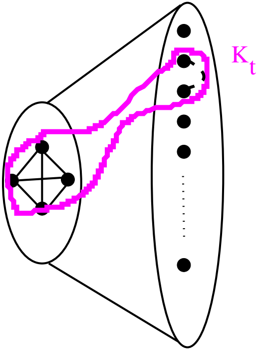
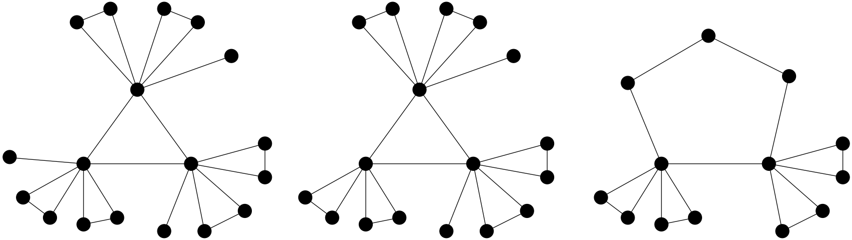


Theorem (Ollmann - '72, Tuza - '86)

$$sat(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor, \quad n \geq 5.$$



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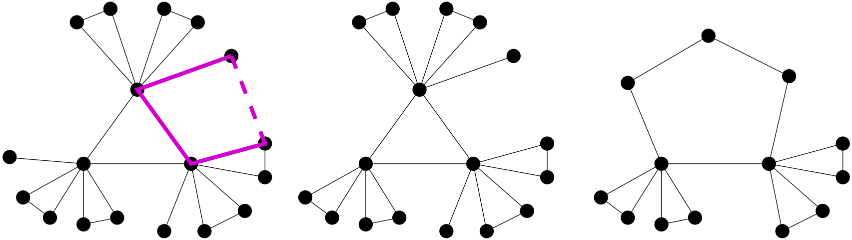

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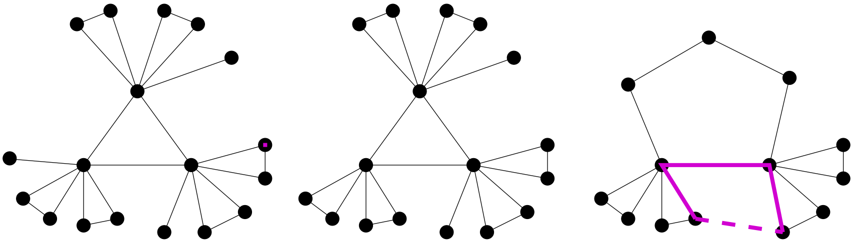

Theorem (Fisher, Fraughnaugh, Langley - '97)

$$sat(n, P_3 - connected) = \lfloor \frac{3n-5}{2} \rfloor.$$

Theorem

$$sat(n, C_5) = \lceil \frac{10n-10}{7} \rceil, n \neq 21.$$

Fisher, Fraughnaugh, Langley, -'95 gave the upper bound.
Y.C.Chen - '09 gave(!) the lower bound.

Problem (FFL)

Determine $sat(n, P_4 - connected)$.

Hamiltonian Cycles

Theorem

$$sat(n, C_n) = \lfloor \frac{3n+1}{2} \rfloor, n \geq 53.$$

Bondy ('72) showed the lower bound. Clark, Entringer, Crane and Shapiro ('83-'86) gave upper bound based on Isaacs' flower snarks (girth 5, 6). L. Stacho ('96) gave further constructions based on the Coxeter graph (girth 7).

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Problem (Horák, Širáň -'86)

Is there a maximally non-hamiltonian graph of girth at least 8?

- **Conjecture (Bollobás - '78)**

  There exist constants, c_1, c_2, such that

$$n + c_1 \frac{n}{l} \leq sat(n, C_l) \leq n + c_2 \frac{n}{l}.$$

- Theorem (Barefoot, Clark, Entringer, Porter, Székely, Tuza - '96)

$$(1 + \frac{1}{2l + 8})n \leq sat(n, C_l)$$

Theorem (Barefoot et al. - '96)

$$sat(n, C_l) \leq (1 + \frac{6}{l-3})n + O(l^2) \textit{ for } l \textit{ odd, } l \geq 9$$

$$sat(n, C_l) \leq (1 + \frac{4}{l-2})n + O(l^3) \textit{ for } l \textit{ even, } l \geq 14$$

Theorem (Barefoot et al. - '96)

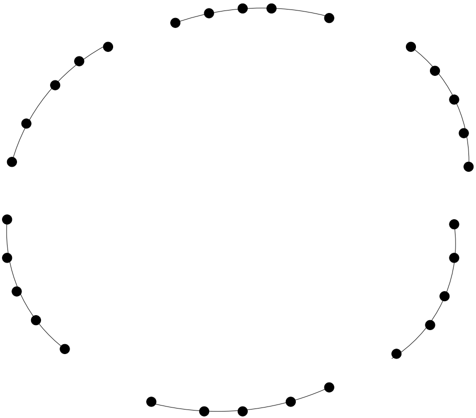
[Gould, Łuczak, S. -'06]
$sat(n, C_l) \leq (1 + \frac{1}{3}\frac{6}{l-3})n + \frac{5l^2}{4}$ *for* l *odd,* $l \geq 9$, $l \geq 17, n \geq 7l$

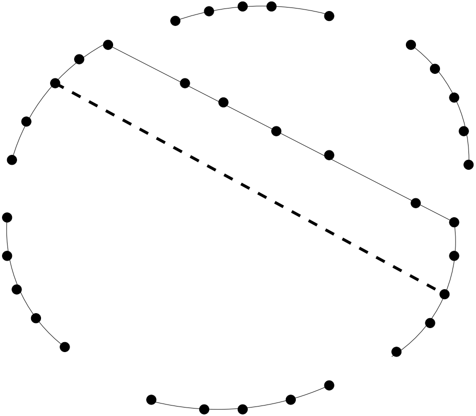
$sat(n, C_l) \leq (1 + \frac{1}{2}\frac{4}{l-2})n + \frac{5l^2}{4}$ *for* l *even* $l \geq 14$, $l \geq 10, n \geq 3l$

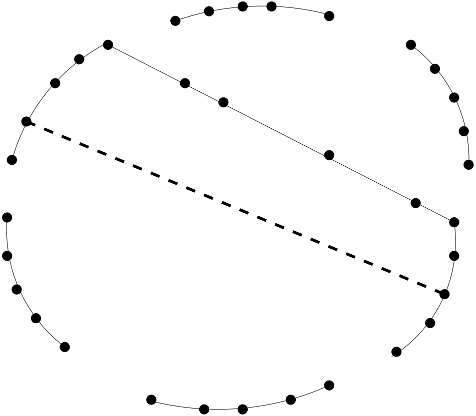
Theorem

[Gould, Łuczak, S. -'06] For $l = 8, 9, 11, 13$ *or* 15 *and* $n \geq 2l$

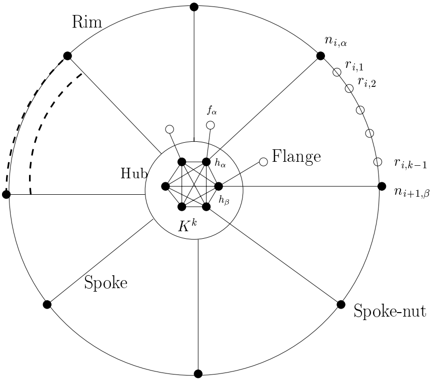
$$sat(n, C_l) \quad < \quad \lceil\frac{3n}{2}\rceil + \frac{l^2}{2}$$







The Even Łuczak Wheel, $l = 2k + 2 \geq 10$



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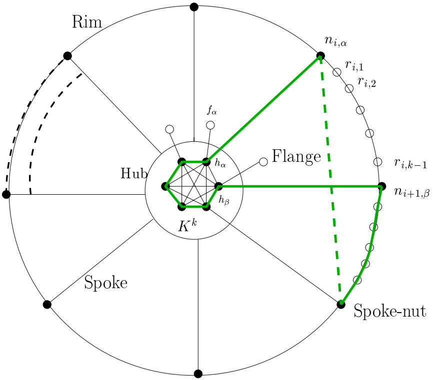

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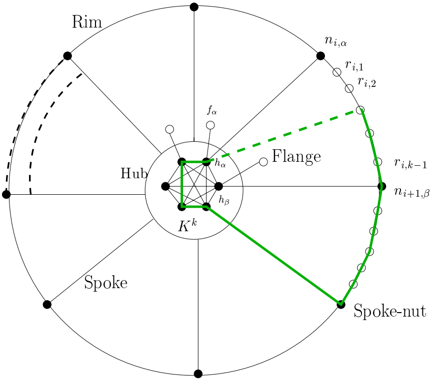

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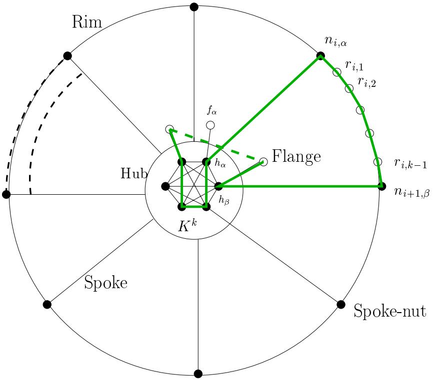


Counting Edges of the Łuczak Wheel

For $l = 2k + 2$ and $n \equiv a \mod k$,

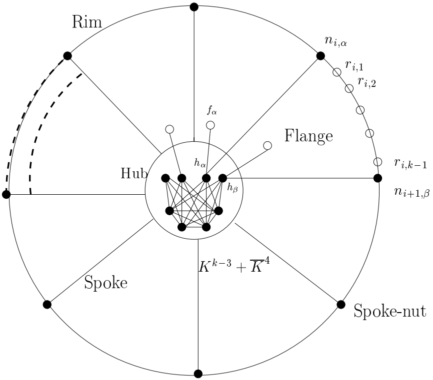
$$|E(L - Wheel)| = \overbrace{(n-k-a)}^{Rim} + \overbrace{\frac{n-k-a}{k}}^{Spokes} + \overbrace{a}^{Flange} + \overbrace{\binom{k}{2}}^{Hub}.$$

Theorem

[GLS] For $k \geq 4$, $l = 2k + 2$, $n \equiv a \mod k$ and $n \geq 3l$,

$$\begin{aligned} sat(n, C_l) &\leq n\left(1 + \frac{1}{k}\right) + \frac{k^2 - 3k - 2}{2} - \frac{a}{k} \\ &\leq n\left(1 + \frac{2}{l-2}\right) + \frac{5l^2}{4}. \end{aligned}$$

The Odd Łuczak Wheel, $l = 2k + 3 \geq 17$



C_l-saturated graphs of minimum size			
l	$sat(n, C_l)$	$n \geq$	Reference
3	$= n - 1$	3	EHM
4	$= \lfloor \frac{3n-5}{2} \rfloor$	5	Ollmann; Tuza
5	$= \lceil \frac{10n-10}{7} \rceil$	21	FFL; Chen
6	$\leq \frac{3n}{2}$	11	Barefoot et al.
7	$\leq \frac{7n+12}{5}$	10	Barefoot et al.
8,9,11,13,15	$\leq \frac{3n}{2} + \frac{l^2}{2}$	$2l$	GLS
≥ 10 and $\equiv 0 \mod 2$	$\leq \left(1 + \frac{2}{l-2}\right) n + \frac{5l^2}{4}$	$3l$	Łuczak wheel
≥ 17 and $\equiv 1 \mod 2$	$\leq \left(1 + \frac{2}{l-3}\right) n + \frac{5l^2}{4}$	$7l$	Łuczak wheel
n	$\lfloor \frac{3n+1}{2} \rfloor$	20	Bondy; CE, CES

Problem (Barefoot et al. - '96)

Determine the value of l which minimizes $sat(n, C_l)$ for fixed n.

Problem

Are any of these constructions optimal? Can one improve the lower bound?

Other Subgraphs

Other values of $sat(n, F)$ known for:

- **matchings** (Mader - '73),
- **stars and paths** (Kászonyi and Tuza - '86),
- **hamiltonian path, P_n** (Frick and Singleton, 05; Dudek, Katona, Wojda - '06)
- **longest path = detour**(Beineke, Dunbar, Frick, '05)
- **disjoint cliques of the same order**(Faudree, Gould, Jacobson, Ferrara - '08)

Difficulties and Hereditary Properties Lacking

Quote from Erdős, Hajnal and Moon:
"One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult."

- $sat(n, F) \not\leq sat(n+1, F)$
- $\mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow sat(n, \mathcal{F}_1) \geq sat(n, \mathcal{F}_2)$
- $F' \subset F \not\Rightarrow sat(n, F') \leq sat(n, F)$

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- $F' \subset F \not\Rightarrow sat(n, F') \leq sat(n, F)$

- $sat(2k-1, P_4) = k+1$ and $sat(2k, P_4) = k$
- $sat(n, \{P_5, S_4\}) = n-1 > sat(n, P_5)$
- $sat(n, K_4) = 2n-3$ but $sat(n, K_5 - S_3) \leq \frac{3}{2}n$

Best known upper bound

Theorem (Kászonyi and Tuza - '86)

Let F be a graph. Set

$$u = |V(F)| - \alpha(F) - 1$$

$$s = min\{e(F') : F' \subseteq F, \alpha(F') = \alpha(F), |V(F')| = \alpha(F) + 1\}.$$

Then

$$sat(n, F) \leq (u + \frac{s-1}{2})n - \frac{u(s+u)}{2}.$$

They considered a clique on u vertices joined to an $(s-1)$-regular graph.

Best Known Lower Bound

Best Known Lower Bound

????

Best Known Lower Bound

Problem

For an arbitrary graph F, determine a non-trivial lower bound on $sat(n, F)$.

Let $sat(n, F, \delta)$ equal *minimum* number of edges in a graph on n vertices and minimum degree δ that is F-saturated.

Theorem (Duffus, Hanson - '86)

$$sat(n, K_3, 2) = 2n - 5, \quad n \geq 5.$$

$$sat(n, K_3, 3) = 3n - 15, \quad n \geq 10.$$

Problem (Bollobás - '95)

Is it true that for every fixed $\delta \geq 1$ *one has* $sat(n, K_3, \delta) = \delta n - O(1)$*?*

Theorem (Gould, S. - '07)

For integers $t \geq 3,\ n \geq 4t-4$,

$$sat(n, K_{t(2)}) \leq sat(n, K_{t(2)}, 2t-3) = \lceil \frac{(4t-5)n - 4t^2 + 6t - 1}{2} \rceil.$$

Problem

Given a fixed graph F, for n sufficiently large determine if the function $sat(n, F, \delta)$ is monotonically increasing in δ.

Theorem (Pikhurko, S. - '08)

There is a constant C such that for all $n \geq 5$ we have

$$2n - Cn^{3/4} \leq sat(n, K_{2,3}) \leq 2n - 3.$$

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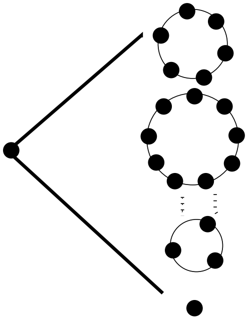

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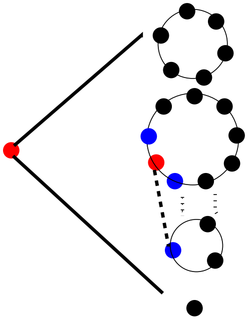


Proof of Lower Bound

Let G be a $K_{2,3}$-saturated graph.

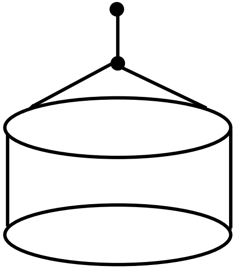
Proof of Lower Bound

Let G be a $K_{2,3}$-saturated graph. If $\delta(G) \geq 4$, then $|E(G)| \geq 2n$ and we are done.

Proof of Lower Bound

Let G be a $K_{2,3}$-saturated graph.

If $\delta(G) = 1$ then,



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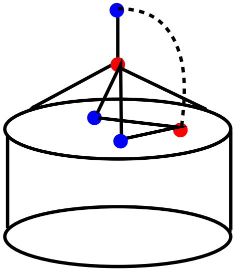

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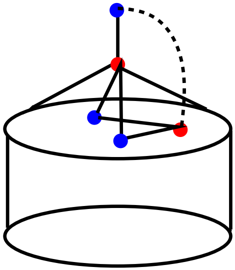
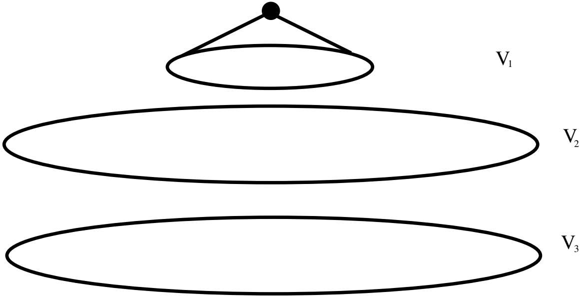


and so $|E(G)| \geq 2n - 3$.

Proof of Lower Bound

Otherwise, $2 \leq \delta(G) \leq 3$, pick vertex of minimum degree and consider breadth-first search tree.



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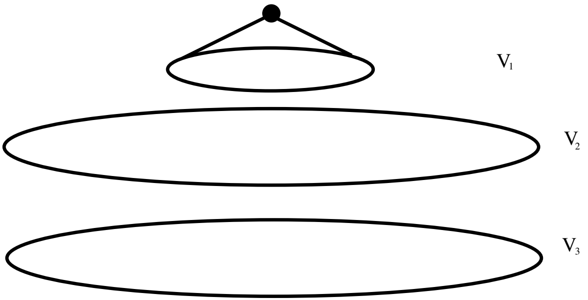
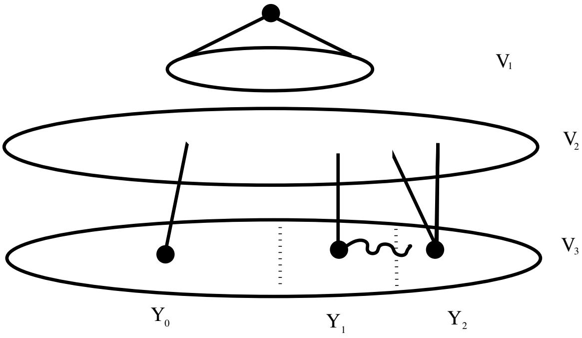


Tree has $n-1$ edges, we must find $n - Cn^{3/4}$ more edges.

Divide and Conquer



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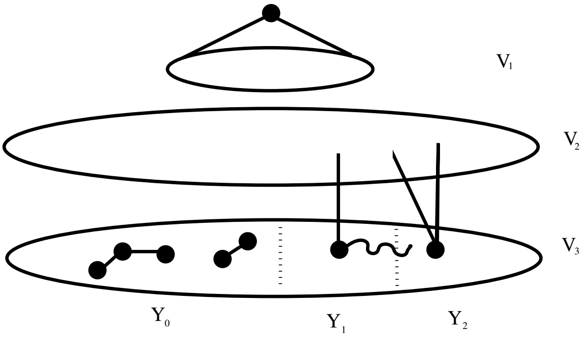
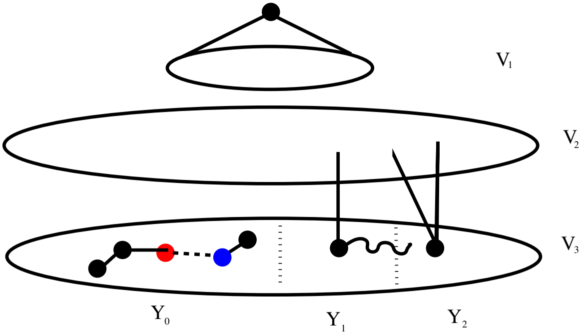
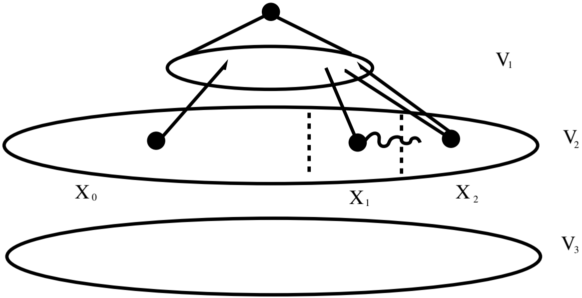


Hit 'em where they're weakest



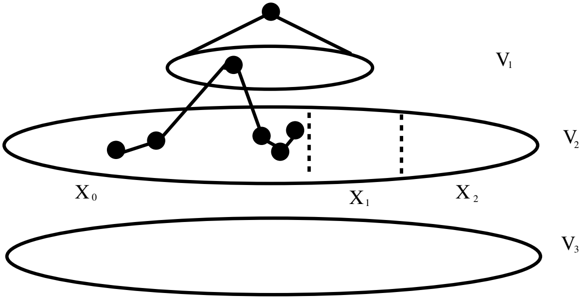
Y_0 has at most one component which is a tree. Pick up an extra $V_3 - 1$ edges.

More Division and More Conquering

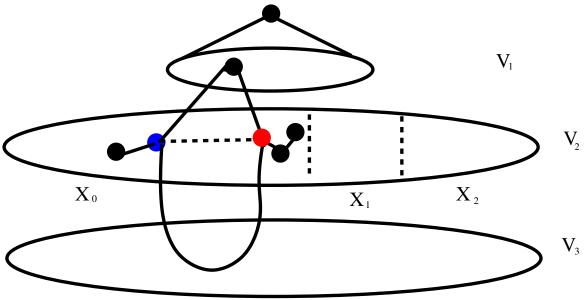


Pick up extra $V_2 - \#(\text{trees} \ \ \text{in} \ X_0)$ edges.

More Division and More Conquering

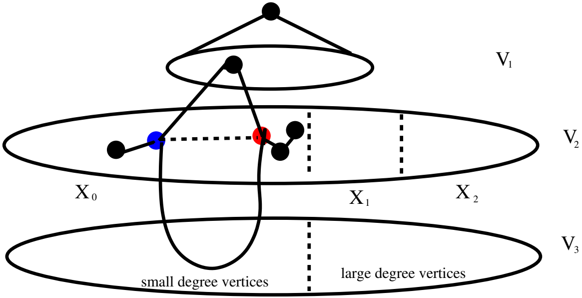


More Hitting Weak Spots



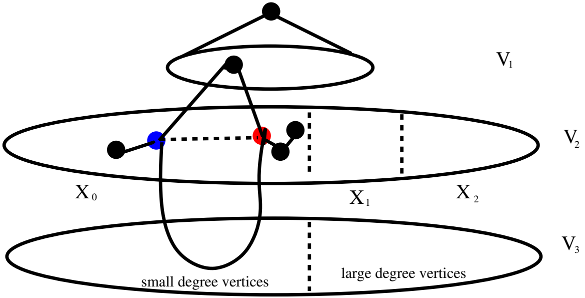
Trees in X_0 are connected via a path of length at most three through V_3.

More Hitting Weak Spots



Small degree vertices can only "serve" so many trees of X_0. So, sum of large degree vertices is large.

More Hitting Weak Spots



Small degree vertices can only "serve" so many trees of X_0. So, sum of large degree vertices is large. This allows us to add #(trees in X_0) $- O(n^{3/4})$ edges to the count. Completes proof. $\square$

Recently, Bohman, Fonoberova and Pikhurko have announced the determination of $sat(n, K_{s_1,s_2,\ldots,s_r})$.

$$sat(n, K_{s_1,s_2,\ldots,s_r}) = (s_1 + s_2 + \ldots + s_{r-1} - 1 + \frac{s_r - 1}{2} + o(1))n,$$

where $s_1 \leq s_2 \leq \ldots \leq s_r$.

And Ramsey Numbers

$F \longrightarrow (F_1, \ldots, F_t)$ if any t coloring of $E(F)$ contains a monochromatic F_i-subgraph of color i for some $i \in [t]$.

Conjecture (Hanson and Toft, '87)

Given $t \geq 2$ and numbers $m_i \geq 3, i \in [t]$, let

$$\mathcal{F} = \{F : F \longrightarrow (K_{m_1}, \ldots, K_{m_t})\}.$$

Let $r = r(K_{m_1}, \ldots, K_{m_t})$ be the classical Ramsey number. Then

$$sat(n, \mathcal{F}) = (r-2)(n-1) - \binom{r-2}{2}.$$

Many Thanks!!

Talk and results are available online at:
http://community.middlebury.edu/∼jschmitt/