Distinct partial sums in cyclic groups

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$(G, +)$ be an abelian group and consider a subset $A \subseteq G$ with $|A| = k$.
Given an ordering $(a_1, \ldots, a_k)$ of the elements of A, define its *partial sums* by $s_0 = 0$ and $s_j = \sum_{i=1}^{j} a_i$ for $1 \leq j \leq k$.

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Conjecture (Alspach, '05)

For any cyclic group $\mathbb{Z}_n$ and any subset $A \subseteq \mathbb{Z}_n \setminus \{0\}$ with $s_k \neq 0$, it is possible to find an ordering of the elements of A such that no two of its partial sums s_i and s_j are equal for $0 \le i < j \le k$.

Bode and Harborth ('05) showed Alspach's Conjecture holds when:

- $|A| = n - 1, n - 2$
- $|A| \leq 5$
- $n \leq 16$ via computer

Conjecture (Archdeacon, '15)

For any cyclic group $\mathbb{Z}_n$ *and any subset* $A \subseteq \mathbb{Z}_n \setminus \{0\}$*, it is possible to find an ordering of the elements of A such that no two of its partial sums* s_i *and* s_j *are equal for* $1 \leq i < j \leq k$*.*

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Archdeacon, Dinitz, Mattern and Stinson ('16):

- Alspach's Conjecture implies Archdeacon's Conjecture
- Archdeacon's Conjecture holds for $|A| \leq 6$
- Verified Archdeacon's Conjecture by computer for $n \leq 25$

Problem

For any cyclic group $\mathbb{Z}_n$ and any positive integer k, what is the smallest order such that from all subsets $A \subseteq \mathbb{Z}_n \setminus \{0\}$ of that order it is possible to find an order of distinct elements of length k that has distinct partial sums?

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Archdeacon, Dinitz, Mattern and Stinson ('16) showed that the smallest order is no more than $2k+1$ via a greedy approach.

Conjecture (Costa, Morini, Pasotti, and Pellegrini '18)

For any abelian group $(G, +)$ *and any subset* $A \subseteq G \setminus \{0\}$ *such that there is no* $x \in A$ *with* $\{x, -x\} \subseteq A$ *and with* $s_k = 0$*, it is possible to find an ordering of the elements of* A *such that no two of its partial sums* s_i *and* s_j *are equal for* $1 \leq i < j \leq k$*.*

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Costa, Morini, Pasotti, and Pellegrini

- When $G = \mathbb{Z}_n$: Costa et al. Conjecture follows immediately from Archdeacon's Conjecture
- Their conjecture holds when $|A| \leq 9$
- Their conjecture holds when $|G| \leq 27$

Motivations

Alspach: cycle decompositions of the complete graph
Archdeacon: embeddings of complete graphs on surfaces to that the faces are 2-colorable and each color class is a k-cycle system
Costa, Morini, Pasotti, and Pellegrini: Heffter systems

Our approach

From now on: $n = p$ is a prime.
For each conjecture and problem we seek an ordering of the elements of A.

To the i^{th}-entry of an ordering of length k associate a variable x_i.

For each conjecture and problem, we construct a polynomial in these k variables over the field of order p.

The set of inputs to these polynomials is $A \times \ldots \times A = A^k$.

Encoding problems by polynomials

For Alspach's Conjecture (encoding):

- $x_i \neq x_j$ for $1 \leq i < j \leq k$ permutation
- $\sum_{\ell=1}^{j} x_\ell \neq 0$ for $1 \leq j < k$ empty partial sum equals 0
- $\sum_{\ell=1}^{i} x_\ell \neq \sum_{\ell=1}^{j} x_\ell$ for $1 \leq i < j \leq k$ distinct partial sums

Encoding problems by polynomials

For Alspach's Conjecture (encoding):

- $x_i - x_j \neq 0$ for $1 \leq i < j \leq k$ permutation
- $\sum_{\ell=1}^{j} x_\ell \neq 0$ for $1 \leq j < k$ empty partial sum equals 0
- $x_i + \cdots + x_j \neq 0$ for $2 \leq i < j \leq k$ distinct partial sums

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Satisfy these simultaneously!

Encoding problems by polynomials

$$F_k := F_k(x_1, \ldots, x_k) = \frac{\prod_{1 \leq i < j \leq k}(x_j - x_i)(x_i + \cdots + x_j)}{(x_1 + \cdots + x_k)}$$

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$$f_k := f_k(x_1, \ldots, x_k) = \prod_{1 \leq i < j \leq k} (x_j - x_i) \prod_{2 \leq i < j \leq k} (x_i + \cdots + x_j)$$

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Theorem (Alon's Non-vanishing Corollary, '99)

Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x_1, \ldots, x_k)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_k]$. Suppose the degree $deg(f)$ of f is $\sum_{i=1}^{k} t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{k} x_i^{t_i}$ in f is nonzero. Then if $A_1, \ldots, A_k$ are subsets of F with $|A_i| > t_i$, there are $a_1 \in A_1, \ldots, a_k \in A_k$ so that $f(a_1, \ldots, a_k) \neq 0$.

A 'low' degree polynomial evaluated over a 'large' box has a 'nonzero'.

$$F_k := F_k(x_1,\ldots,x_k) = \prod_{1\le i<j\le k} (x_j - x_i)(x_i + \cdots + x_j) \ / \ (x_1 + \cdots + x_k)$$

degree of F_k is $2\binom{k}{2} - 1 = k(k-1) - 1$

Monomials of choice: $m_{k,j} = c_{k,j} x_1^{k-1} \cdots x_j^{k-2} \cdots x_k^{k-1}$ for $1 \le j \le k$

Coefficient $c_{k,j}$ on $m_{k,j}$ computed over $\mathbb{Z}$

$k \setminus j$	1	2	3	4	5
2	1				
3	-1	0			
4	1	-1			
5	4	-2	-4		
6	-28	-40	-20		
7	966	1662	1338	0	
8	-366468	-92412	144324	314556	
9	-359616276	-130597656	72122706	254703096	326776260
10	595372941856	1404671795722	1785841044600	1435120776421	546395688803

$$c_{k,j} = \pm c_{k,k+1-j} \text{ for all } k \text{ and } 1 \leq j \leq \lfloor \tfrac{k}{2} \rfloor$$

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$c_{k,j} = \pm c_{k,k+1-j}$ for all k and $1 \leq j \leq \lfloor \frac{k}{2} \rfloor$
Compute prime factorization of each integer in the table.
e.g.:
$k = 7; 966 = 2 \times 3 \times 7 \times 23; 1662 = 2 \times 3 \times 277; 1338 = 2 \times 3 \times 223; 0$

Theorem (Hicks, Ollis, S.)

Alspach's Conjecture is true for prime n and $k \leq 10$.

Corollary

Archdeacon's Conjecture and Costa et al. Conjecture (for $G = \mathbb{Z}_n$*) is true for prime n and* $k \leq 10$.

Theorem

Let p be prime. From any subset of size $2k-3$ of $\mathbb{Z}_p \setminus \{0\}$ we can construct a sequence of length k with distinct partial sums.

PROOF: The coefficient on

$$x_1^{k-1} x_2^0 x_3^2 x_4^4 \cdots x_k^{2k-4}$$

of

$$f_k(x_1, x_2, \ldots, x_k) = \prod_{1 \le i < j \le k} (x_j - x_i) \prod_{2 \le i < j \le k} (x_i + \cdots + x_j)$$

is $(-1)^{k-1}$.

Working towards a theoretical result

$$f_k(x_1, x_2, \ldots, x_k) = \prod_{1 \le i < j \le k} (x_j - x_i) \prod_{2 \le i < j \le k} (x_i + \cdots + x_j).$$

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Same argument: find a leading monomial with nonzero coefficient and degree of the highest term $2k - d - 1$.
Then Alon's Non-vanishing Corollary implies a solution to Problem when $|A| = 2k - d$ (for all but finitely many prime values of p).

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Then Alon's Non-vanishing Corollary implies a solution to Problem when $|A| = 2k - d$ (for all but finitely many prime values of p).

Monomial we choose is

$$x_1^{k-1} x_2^0 x_3^2 x_4^4 \cdots x_{k-d+2}^{2k-2d} x_{k-d+3}^{2k-d-1} x_{k-d+4}^{2k-d-1} \cdots x_k^{2k-d-1}.$$

Theorem (Hicks, Ollis, S.)

Fix $d \in \mathbb{N}$ with $d > 3$ and let $k \geq \min(8, d^2/8)$. For almost all primes p there are at most $(d-3)(d-2)(d-1)/6$ values of k (with $k \geq d^2/8$) where it is not the case that we can construct a sequence of k elements with distinct partial sums from any set of $2k - d$ distinct elements of $\mathbb{Z}_p \setminus \{0\}$.

Thanks!