Degree Sum Conditions in Graph Pebbling

John Schmitt
Middlebury College

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AMS-MAA Joint Meetings

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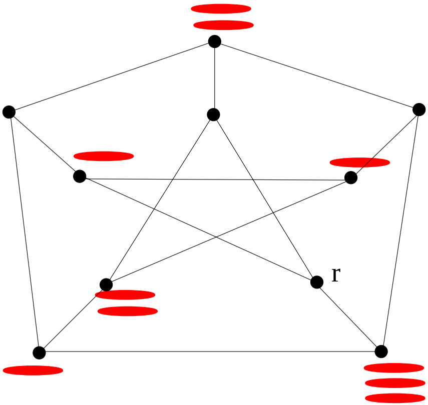
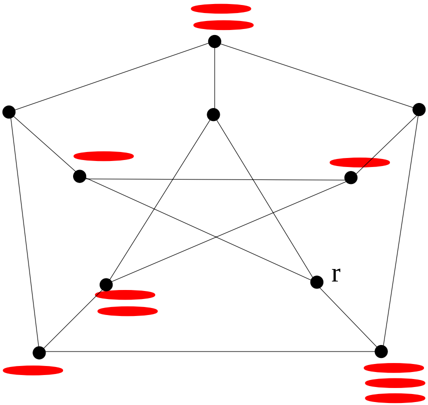
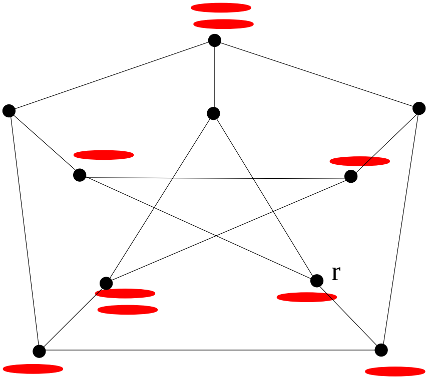
joint work with Anna Blasiak
(Middlebury College '07 - now at Cornell U.)



Graph pebbling is a mathematical model for the transmission of consumable resources.



A *pebbling move* consists of removing two pebbles from a vertex and placing one of them on an adjacent vertex.



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Pebbling number of a graph G, denoted $\pi(G)$, is the least number of pebbles necessary to guarantee that, regardless of distribution of pebbles and regardless of target vertex, there exists a sequence of pebbling moves that enables us to place a pebble on the target vertex.

Lower Bound on the Pebbling Number

$$\pi(G) \geq \max\{n, 2^{diam(G)}\}$$

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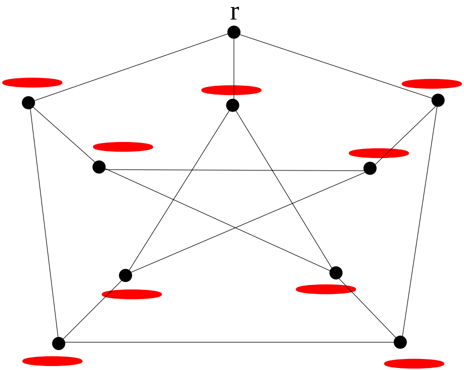

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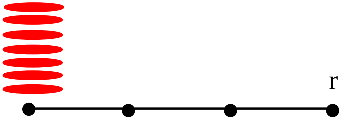

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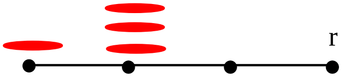

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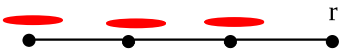

If $\pi(G) = n$ then G is called *Class 0*.

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Class 0 graphs include:

- the complete graph, K_n
- the complete t-partite graph (except stars), $K_{p_1,p_2,\ldots,p_t}$
- the d-dimensional hypercube, Q_d (F. Chung, '92)
- Petersen graph
- Kneser graph (certain instances)

Problem (Hurlbert):

Find necessary and sufficient conditions for G to be Class 0.

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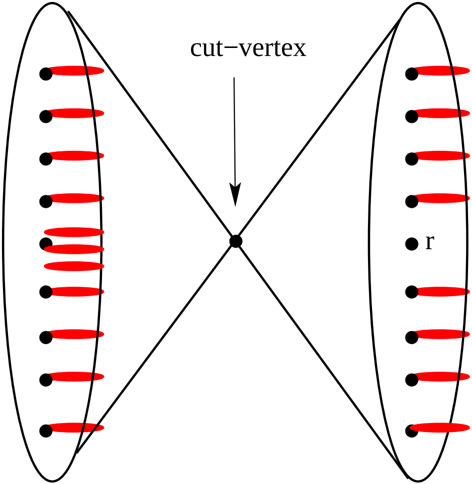
Most results have placed conditions on diameter and/or connectivity.

Class 0 graphs have connectivity at least two



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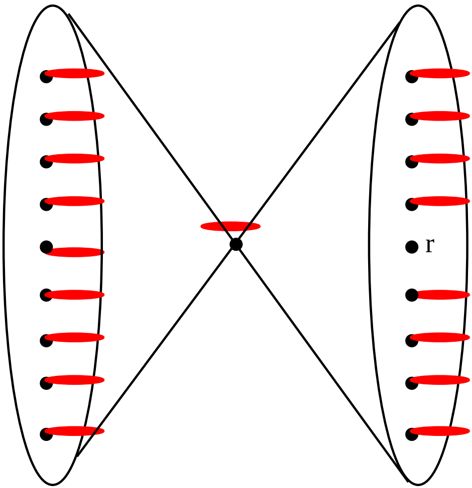

Degree Sum Result

Theorem (Blasiak, S.) If for any pair of non-adjacent vertices, u, v, in G we have $d(u) + d(v) \geq n$ then G is Class 0.

Corollary (Czygrinow, Hurlbert - '03) If $\delta(G) \geq \lceil \frac{n}{2} \rceil$ then G is Class 0.

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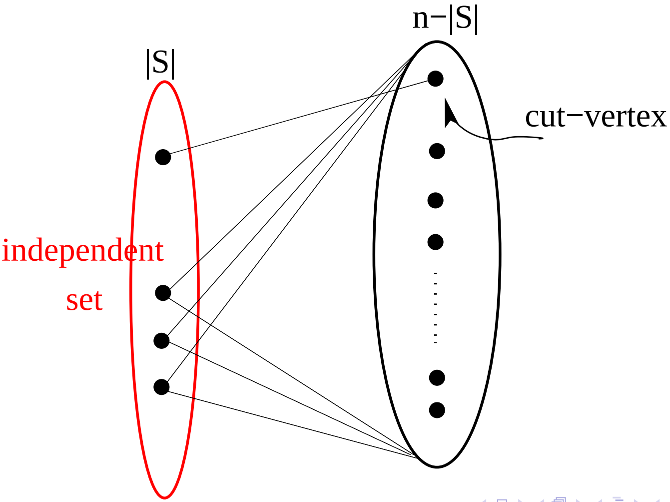
Theorem (Blasiak, S.) Let G be a graph on $n \geq 6$ vertices. If for each maximal independent set, S, of G we have

$$\sum_{v \in S} d(v) \quad \geq \quad (|S| - 1)(n - |S|) + 2$$

then G is Class 0.

Showing sharpness



The proof...

The proof relies on a result of Clarke, Hochberg, Hurlbert ('97) that characterizes graphs with diameter two and connectivity at least two which are not Class 0.

We show that our degree sum condition implies that G has diameter two and is 2-connected. The graphs characterized by CHH do not meet the degree sum condition and so any graph meeting the degree sum condition must be Class 0.

Edge density for Class 0

Corollary (Pachter, Snevily, Voxman - '95) Let G be a connected graph with $n \geq 6$ vertices and $|E(G)|$ edges. If $|E(G)| \geq \binom{n-1}{2} + 2$, then G is Class 0.

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Proof: As $\binom{n}{2} - (\binom{n-1}{2} + 2) = n - 3$, the hypothesis implies that G has at most $n - 3$ non-edges.

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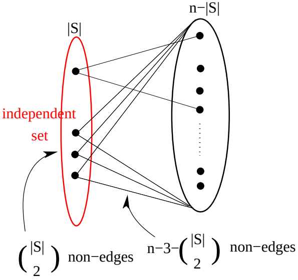

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Proof: Thus,

$$\begin{aligned} \sum_{v \in S} d(v) &\geq |S|(n - |S|) - (n - 3 - \binom{|S|}{2}) \\ &\geq (|S| - 1)(n - |S|) + 2. \end{aligned}$$

Applying the theorem, G is Class 0.

On-going work

We are currently investigating the minimum number of edges in a Class 0 graph .... look for more soon!