Extremal Problems on Bipartite Graphs

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An Erdős problem in number theory, 1938

Problem

How many integers can one choose between 1 and n so that no one divides the product of two others?

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Suppose $n = 100$.
A bad set: $S = \{6, 11, 13, 22, 33\}$, since 6 divides 22×33.

An Erdős problem in number theory, 1938

Problem

How many integers can one choose between 1 and n so that no one divides the product of two others?

Suppose $n = 100$.

A good set

$$S = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$$

A solution

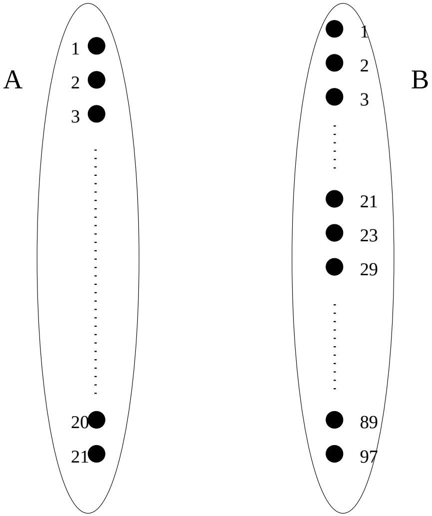
Let $A = \{x : 1 \leq x \leq n^{2/3}\}$, $P = \{p : p \text{ } prime, n^{2/3} < p \leq n\}$, $B = A \cup P$
Each integer between 1 and n can be written as a product of two integers, one from A and one from B.

A solution

Let $A = \{x : 1 \le x \le n^{2/3}\}$, $P = \{p : p \ prime, n^{2/3} < p \le n\}$,
$B = A \cup P$
Each integer between 1 and n can be written as a product of two integers, one from A and one from B.
If $n = 100$, we have, $A = \{1, 2, 3, \ldots, 21\}$
$P = \{23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$

$$\begin{aligned} B &= \{1, 2, 3, \ldots, 21, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, \\ &\quad 73, 79, 83, 89, 97\} \end{aligned}$$

Turning it into a graph problem



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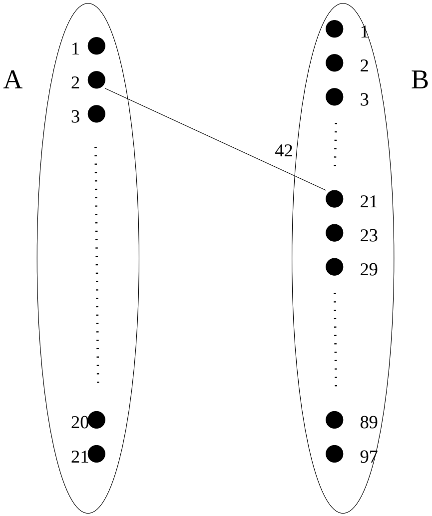

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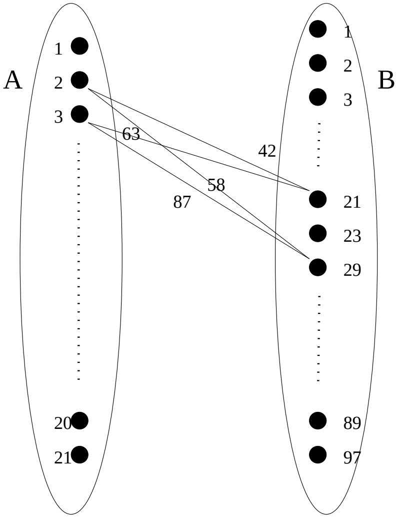
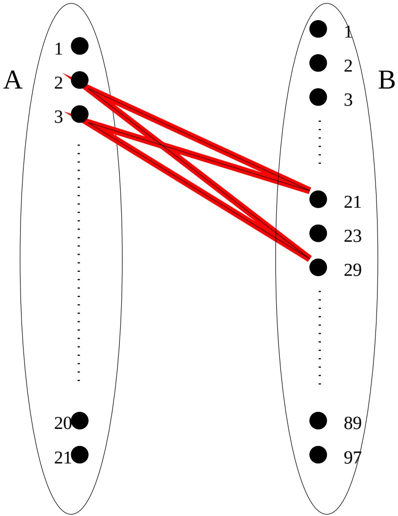


(58)(63)=3654

(42)(87)=3654

Turning it into a graph problem



FORBID QUADRILATERALS

FORBID C_4 = K_{2,2}

Erdős determined that the maximum number of edges in a $K_{2,2}$(quadrilateral)-free balanced bipartite graph with $|A| = k = |B|$ is $3k^{3/2}$.

This implies that the answer to his question is

$$\pi(n) + O(n^{3/4}),$$

where $\pi(n)$ denotes the number of primes less than or equal to n.

You can't do much better than choosing the primes.

A construction of a $K_{2,2}$-free graph with many edges

Consider a finite projective plane of order p: a set of $p^2 + p + 1$ points together with a collection of $\{p + 1\}$-subsets of these points (called lines) which has the following properties -

- each line is incident with $p + 1$ points,
- each point is incident with $p + 1$ lines,
- any pair of points are joined by exactly one line,
- any pair of lines intersect in exactly one point.

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Build the point-line incidence graph G of a projective plane of order p. This is a $p+1$-regular bipartite graph which has $2(p^2 + p + 1) = n$ vertices and $O(n^{3/2})$ edges.

Showing the upper bound

Let $G_{n,n}$ be a balanced bipartite $K_{2,2}$-free graph, then $G_{n,n}$ contains at most $\frac{n}{2}(1+\sqrt{4n-3})$ edges.

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If we denote the degrees of the vertices in the first partite set by $d_1, \ldots, d_n$, then it follows that

$$\binom{n}{2} \geq \Sigma \binom{d_i}{2},$$

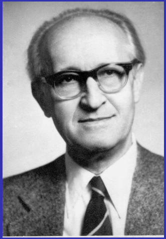
since each pair of vertices in the second set has at most one common neighbor in the first set.

If $G_{n,n}$ contains more edges than stated then after some algebraic manipulation using this inequality we may arrive at a contradiction.

The tip of the iceberg

Extremal Graph Theory deals with the inevitable occurrence of some specified structure when some graph invariant (such as the edge density) exceeds a certain threshold.

The father of Extremal Graph Theory

Paul Turán 

Turán's Theorem, 1941 The maximum number of edges in K_t-free graph on n vertices, $ex(n, K_t)$, is

$$(1 - \frac{1}{t-1})\frac{n^2}{2}.$$

Furthermore, the only graph which achieves this bound is $K_{n_1,n_2,\ldots n_{t-1}}$, where the n_i are as balanced as possible.

Erdős-Stone-Simonovits Theorem

Theorem Given a graph F with chromatic number, $\chi(F)$, at least three, the maximum number of edges in an F-free graph on n, $ex(n, F)$, vertices is

$$(1 - \frac{1}{\chi(F) - 1})\frac{n^2}{2} + o(n^2).$$

The problem of Zarankiewicz

The Erdős-Stone-Simonovits Theorem does not apply to graphs with chromatic number two!

Problem Determine $ex(n, K_{s,t})$.

Problem of Zarankiewicz Determine $ex(n, n; K_{s,t})$, the maximum number of edges in a $K_{s,t}$-free bipartite graph, with partite sets of size n.

Obtaining an upper bound for the latter problem yields an upper bound for the former.

It can be shown that for $2 \leq t \leq s$ we have

$$ex(n, n; K_{s,t}) < cn^{2-1/t},$$

where c is a constant depending on s and t.

Is it true that there exists a constant c' depending on s and t such that $ex(n, n; K_{s,t}) > c'n^{2-1/t}$?

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Yes for $K_{2,2}$ (Erdős) and $K_{s,2}$ (Erdős, Rényi and Sós)

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Yes for $K_{3,3}$ (Brown, and later Füredi)

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where c is a constant depending on s and t.
Is it true that there exists a constant c' depending on s and t such that $ex(n, n; K_{s,t}) > c'n^{2-1/t}$?

Yes for $K_{\geq t!+1,t}$ (Kollár, Rónyai and Szabó)

A variation of the Zarankiewicz problem

Let $S = (a_1, a_2, \ldots, a_m; b_1, b_2, \ldots b_n)$ be a pair of positive integer sequences.
We say that S is a *bigraphic pair* if there exists some simple bipartite graph G with partite sets $X = \{x_1, \ldots x_m\}$ and $Y = \{y_1, \ldots y_n\}$ such that the degree of x_i is a_i and the degree of y_j is b_j.

A variation of the Zarankiewicz problem

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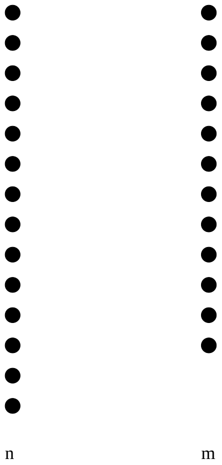
Let $\sigma(n, m; F)$ denote the minimum sum of one of these pairs which guarantees the existence of some G containing F as a subgraph.
Problem Determine $\sigma(n, m; K_{s,t})$.

Our result

Theorem (Ferrara, Jacobson, S., Siggers) For all $1 \leq s \leq t$, there exists an m_0 such that for $n \geq m \geq m_0$ the following holds:

$$\sigma(n, m; K_{s,t}) = n(s-1) + m(t-1) - (t-1)(s-1) + 1.$$

Proof of Lower Bound



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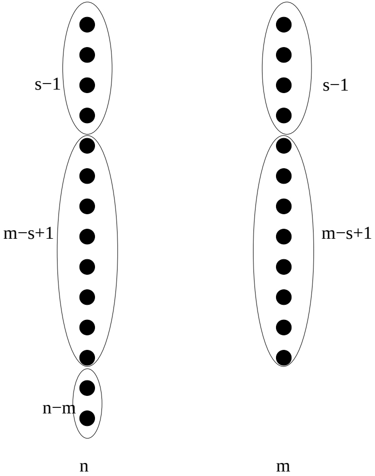

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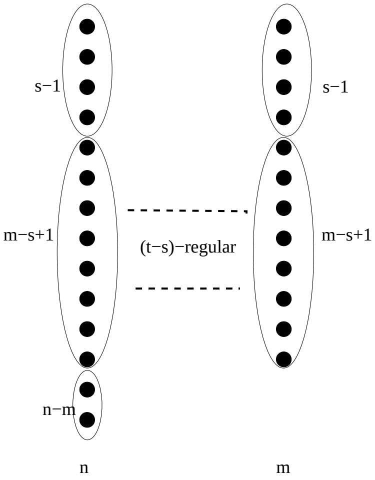

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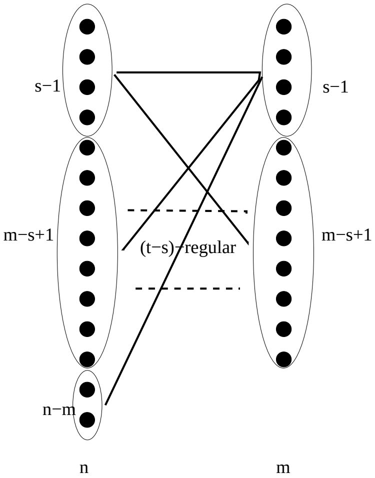

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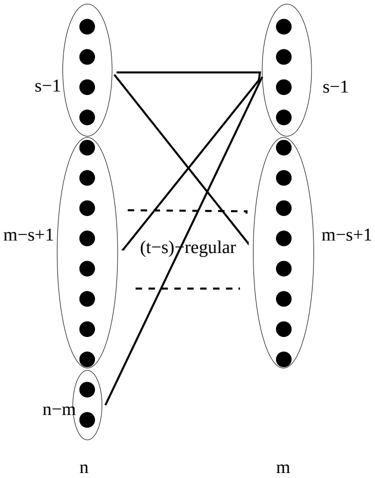
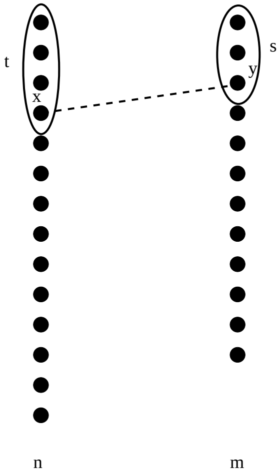


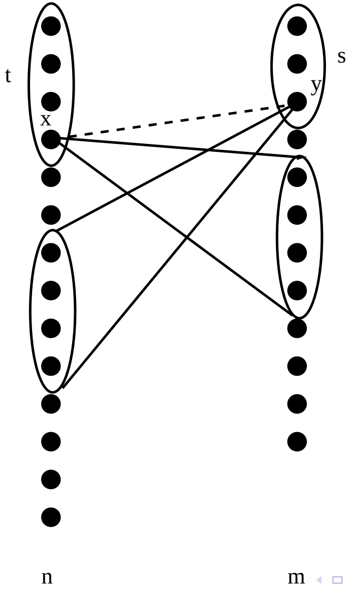
$S = (m^{s-1}, (t-1)^{m-s+1}, (s-1)^{n-m}; n^{s-1}, (t-1)^{m-s+1})$ and the sum of the first sequence (and the second) is
$n(s-1) + m(t-1) - (t-1)(s-1)$.

Proof of Upper Bound

Let S be a bigraphic pair with at least the sum given. Start by picking the best realization of S on the vertices of highest degree.



ab is an edge



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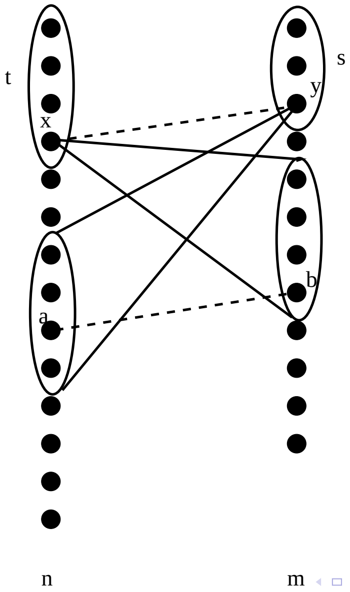

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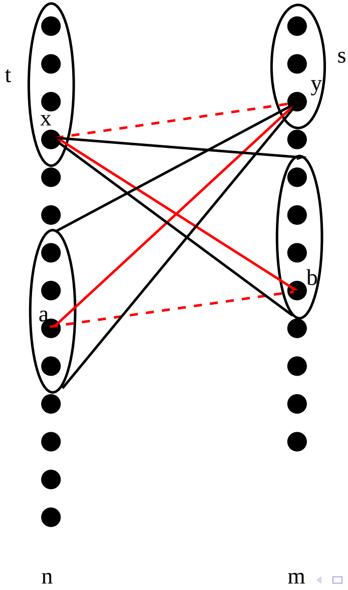

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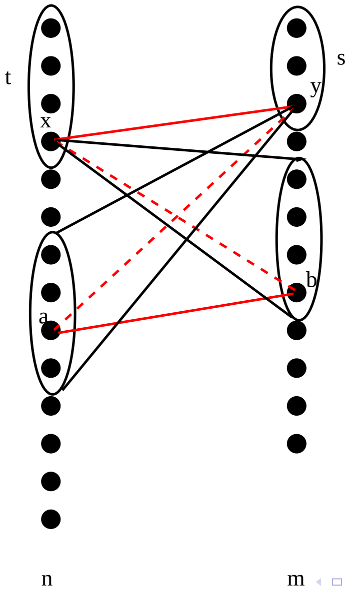

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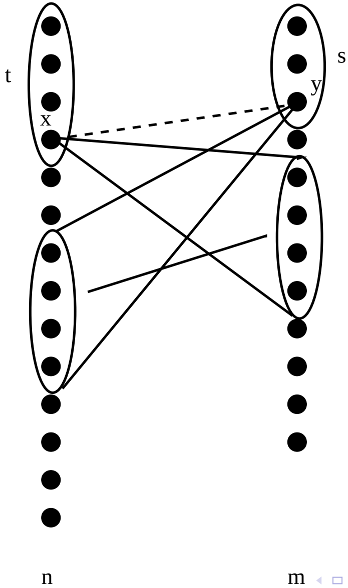
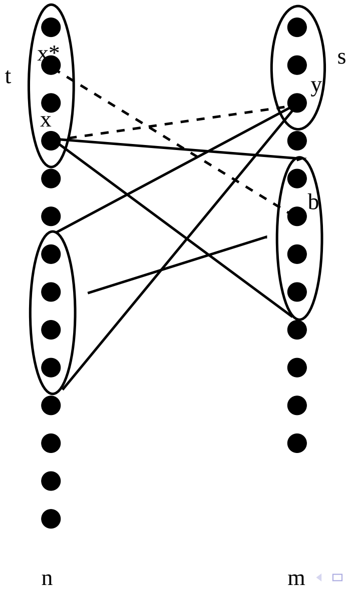
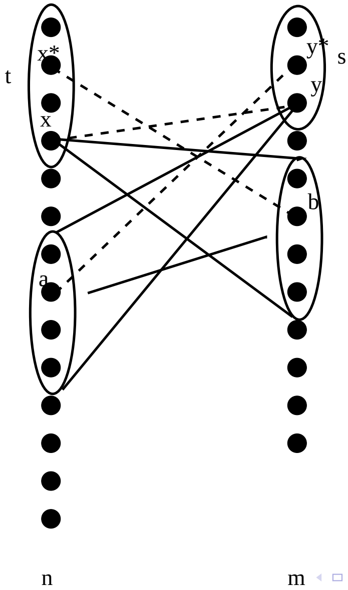
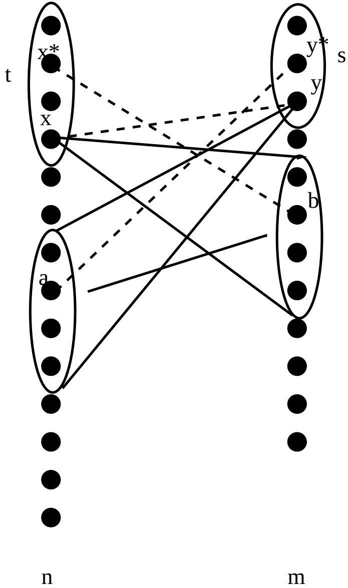


x^*b is not an edge as otherwise $d(b) > d(y)$



y^*a is not an edge as otherwise $d(a) > d(x)$

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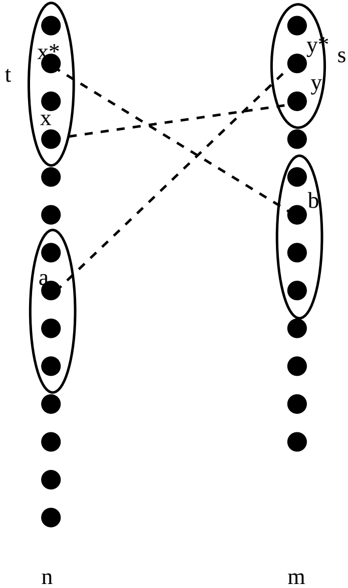
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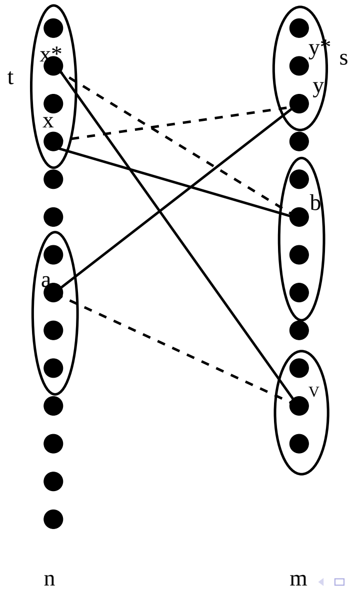

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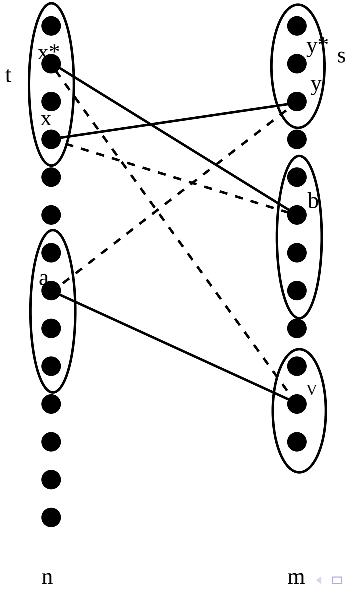
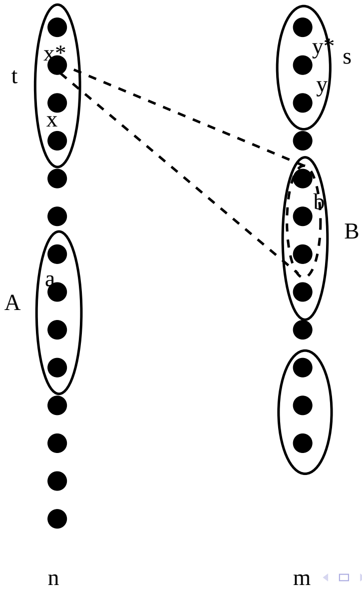


$|A|, |B| < (s-1)(t-1)+1$ (i.e. these sets are small)



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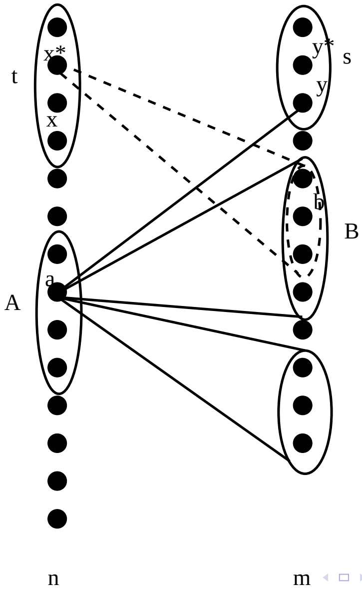
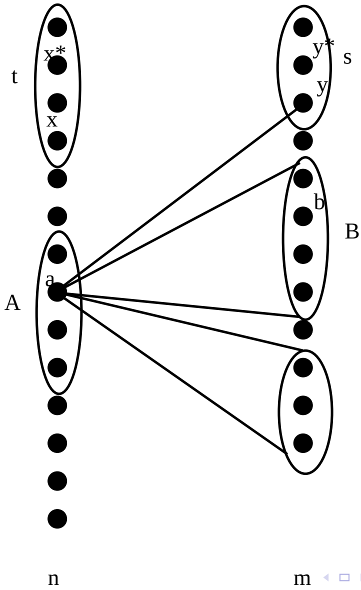
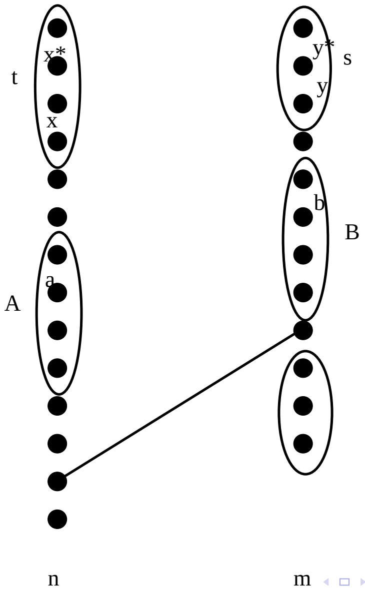


$d(x^*), d(y^*)$ are small as otherwise $d(a) > d(x)$



Finally....



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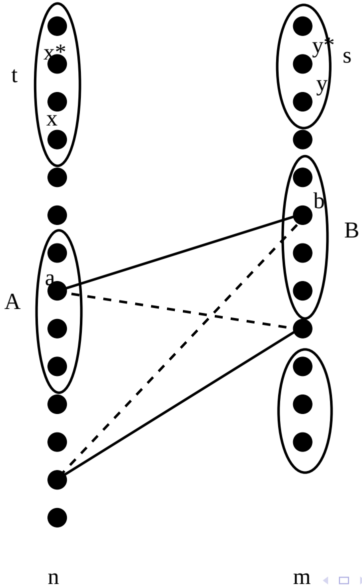

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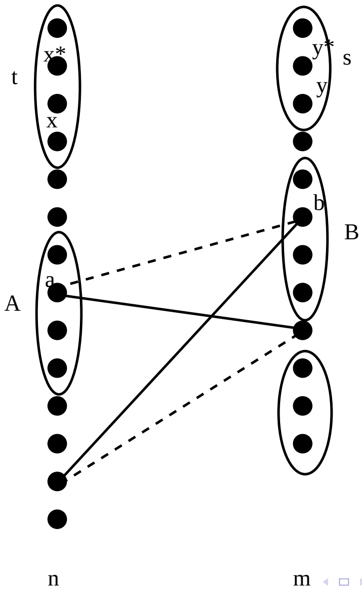

Further Results

Lemma (FJSS)

If $n \geq m$ *then* $\sigma(n, m; P_3) = m + 1, \sigma(n, m : P_4) = n + 1.$

Theorem (FJSS)

For $t \geq 2$ *and* $n \geq m \geq t + 1$,

$$\sigma(n, m : P_{2t+1}) = \sigma(n, m : P_{2t+2}) = n(t-1) + m - (t-1) + 1.$$

Theorem (FJSS)

For $t \geq 2$ *and* $n \geq m \geq 2(t + 1)$,

$$\sigma(n, m : C_{2t}) = n(t-1) + m - (t-1) + 1.$$

Open Problems

- Determine $\sigma(n, m; F)$ for any bipartite graph F.
- Solve the Zarankiewicz problem, somebody - please!