Minimum Size of Bipartite-Saturated Graphs

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joint work with
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July, 2007
British Combinatorics Conference
(Reading, UK)

Definition

A graph G is ***F*-saturated** if

$F \not\subset G$,

$F \subset G + e$ for any $e \in E(\overline{G})$.

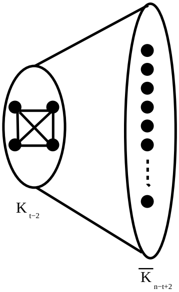
Problem

Determine the minimum number of edges, $sat(n, F)$*, of an* F*-saturated graph.*

Theorem (Erdős, Hajnal, Moon - 1964)

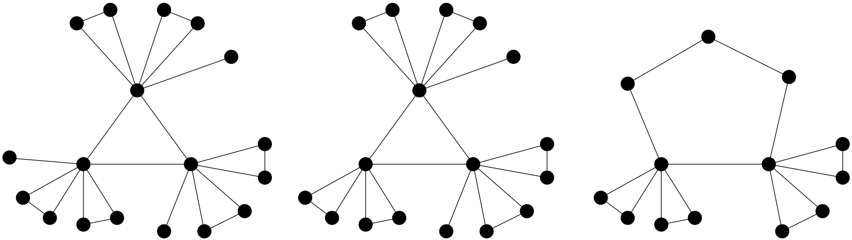
$$sat(n, K_t) = (t-2)(n-1) - \binom{t-2}{2}$$

Furthermore, the only K_t-saturated graph with this many edges is $K_{t-2} + \overline{K}_{n-t+2}$.



Theorem (Ollmann - '72, Tuza - '86, Fisher, Fraughnaugh, Langley -'97)

$$sat(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor, \quad n \geq 5$$



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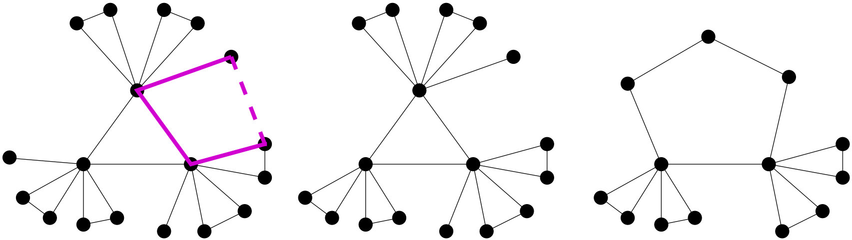


Theorem (Y.C.Chen, 07+)

$$sat(n, C_5) = \lceil \frac{10n-10}{7} \rceil, n \geq 21$$

Hamiltonian Cycles

Theorem

$$sat(n, C_n) = \lfloor \frac{3n+1}{2} \rfloor, n \geq 53$$

Bondy ('72) showed the lower bound. Clark, Entringer, Crane and Shapiro ('83-'86) gave upper bound based on Isaacs' flower snarks (girth 5, 6). L. Stacho ('96) gave further constructions based on the Coxeter graph (girth 7).

Problem (Horák, Širáň -'86)

Is there a maximally non-hamiltonian graph of girth at least 8 (that meets this bound)?

- **Conjecture (Bollobás - '78)**

$$n + c_1\frac{n}{l} \leq sat(n, C_l) \leq n + c_2\frac{n}{l}$$

- Theorem (Barefoot, Clark, Entringer, Porter, Székely, Tuza - '96)

$$(1 + \frac{1}{2l+8})n \leq sat(n, C_l)$$

Theorem (Barefoot et al. - '96)

$$sat(n, C_l) \leq (1 + \frac{6}{l-3})n + O(l^2) \text{ for } l \text{ odd, } l \geq 9$$

$$sat(n, C_l) \leq (1 + \frac{4}{l-2})n + O(l^3) \text{ for } l \text{ even, } l \geq 14$$

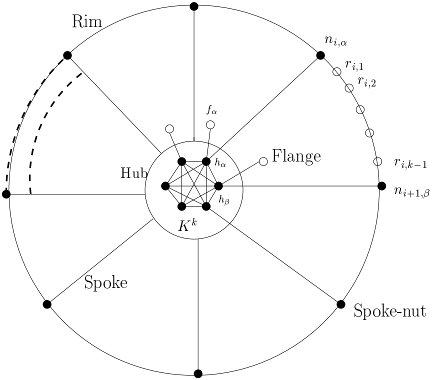
Theorem (Barefoot et al. - '96)

[Gould, Łuczak, S. -'06]

$sat(n, C_l) \leq (1 + \frac{1}{3}\frac{6}{l-3})n + \frac{5l^2}{4}$ *for l odd,* $l \geq 9$, $l \geq 17, n \geq 7l$

$sat(n, C_l) \leq (1 + \frac{1}{2}\frac{4}{l-2})n + \frac{5l^2}{4}$ *for l even* $l \geq 14$, $l \geq 10, n \geq 3l$

The Even Łuczak Wheel, $l = 2k + 2 \geq 10$



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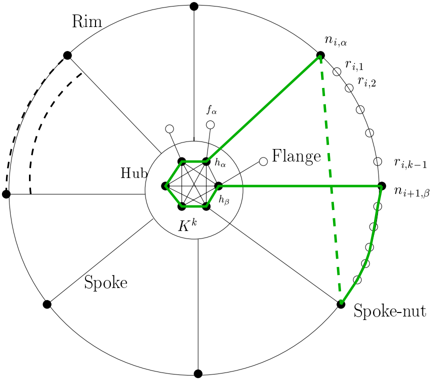

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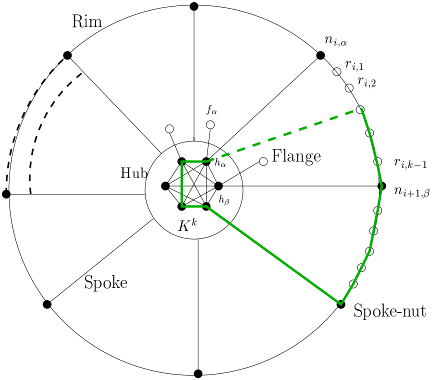


Other Subgraphs

Other values of $sat(n, F)$ known for:

- **matchings** (Mader - '73),
- **paths and stars** (Kászonyi and Tuza - '86),
- **hamiltonian path, P_n** (Frick and Singleton, 05; Dudek, Katona, Wojda - '06)

$$sat(n, P_n) = \lceil \frac{3n-2}{2} \rceil, n \geq 54$$

- **longest path = detour**(Beineke, Dunbar, Frick, '05)

Difficulties and Hereditary Properties Lacking

Quote from Erdős, Hajnal and Moon:
"One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult."

- $sat(n, F) \not\leq sat(n+1, F)$
- $\mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow sat(n, \mathcal{F}_1) \geq sat(n, \mathcal{F}_2)$
- $F' \subset F \not\Rightarrow sat(n, F') \leq sat(n, F)$

Best known upper bound

Theorem (Kászonyi and Tuza)

Let F be a graph. Set

$$u := u(F) = |V(F)| - \alpha(F) - 1$$

$$s := s(F) = min\{e(F') : F' \subseteq F, \alpha(F') = \alpha(F), |V(F')| = \alpha(F)+1\}.$$

Then

$$sat(n, F) \leq (u + \frac{s-1}{2})n - \frac{u(s+u)}{2}.$$

They considered a clique on u vertices joined to an $(s-1)$-regular graph.

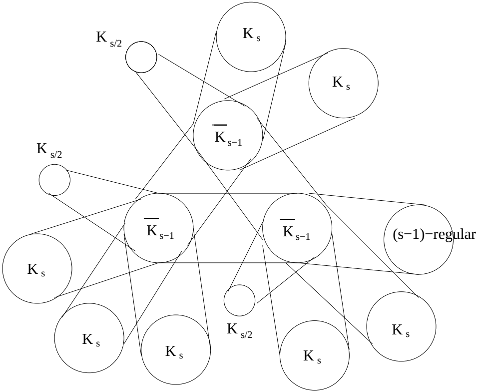
Best Known Lower Bound

????

Problem

For an arbitrary graph F, determine a non-trivial lower bound on $sat(n, F)$.

Saturation for Bipartite Graphs, $K_{s,s}$



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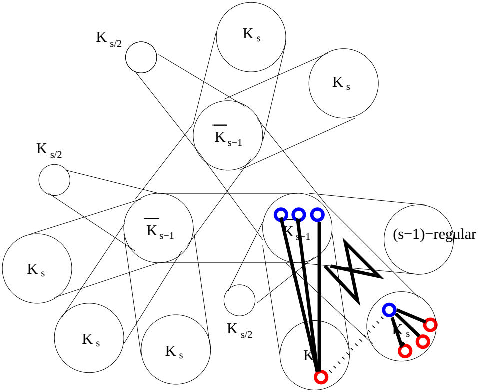

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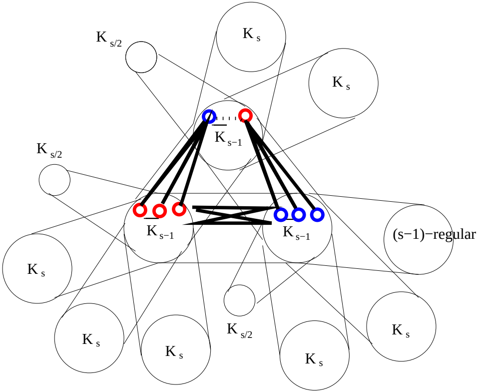

Yields an improvement of a constant times s^2 over the KT bound.

Theorem (O.Pikhurko, S.)

There is a constant C such that for all $n \geq 5$ we have

$$2n - Cn^{3/4} \leq sat(n, K_{2,3}) \leq 2n - 3.$$

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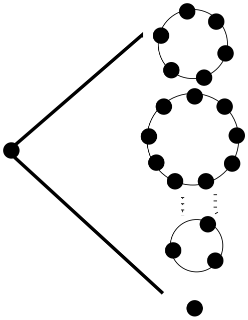

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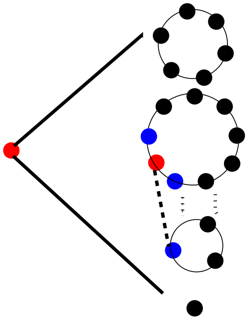

Proof of Lower Bound

Let G be a $K_{2,3}$-saturated graph.

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Let G be a $K_{2,3}$-saturated graph. If $\delta(G) \geq 4$, then $|E(G)| \geq 2n$ and we are done.

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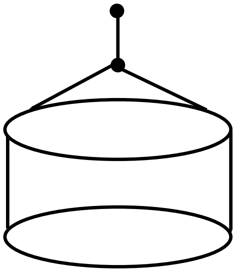
If $\delta(G) = 1$ then,



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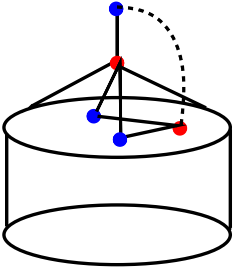

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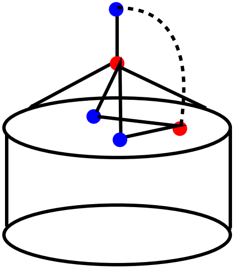
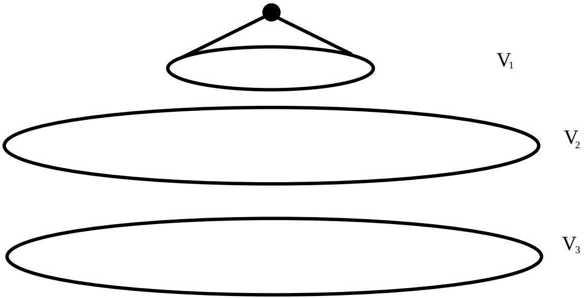


and so $|E(G)| \geq 2n - 3$.

Proof of Lower Bound

Otherwise, $2 \leq \delta(G) \leq 3$, pick vertex of minimum degree and consider breadth-first search tree.



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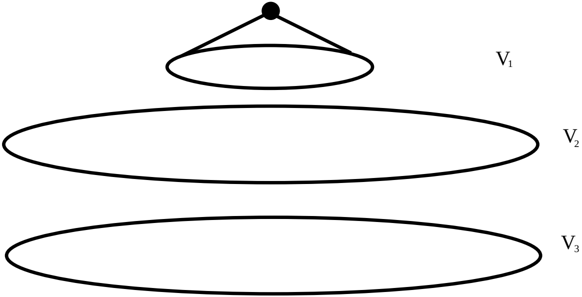
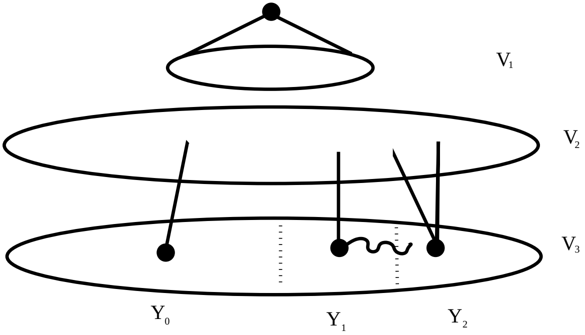


Tree has $n - 1$ edges, we must find $n - Cn^{3/4}$ more edges.

Divide and Conquer



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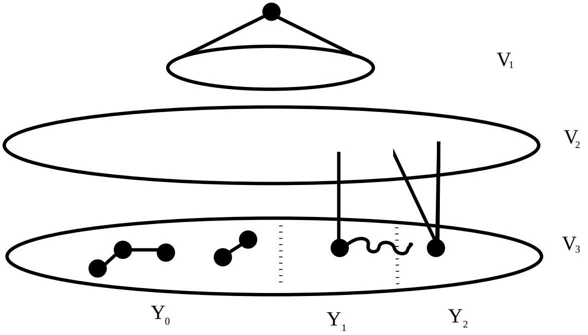
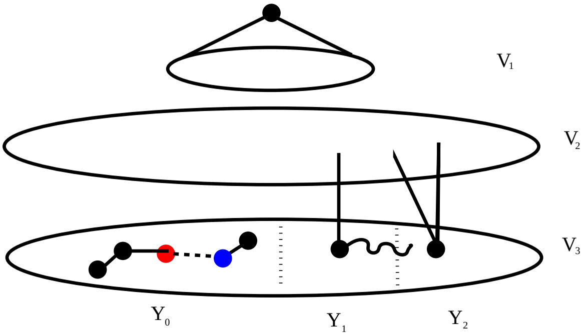
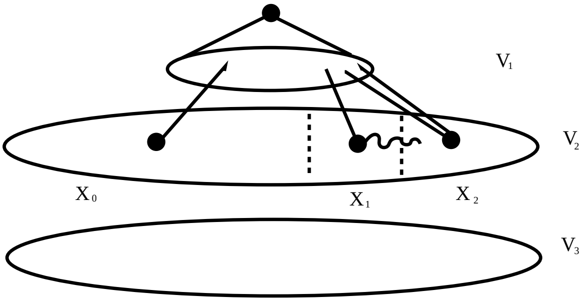


Hit 'em where they're weakest



Y_0 has at most one component which is a tree. Pick up an extra $V_3 - 1$ edges.

More Division and More Conquering



Pick up extra $V_2 - \#(\text{trees} \ \text{ in } X_0)$ edges.

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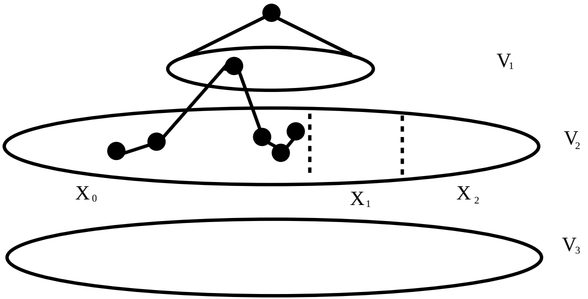
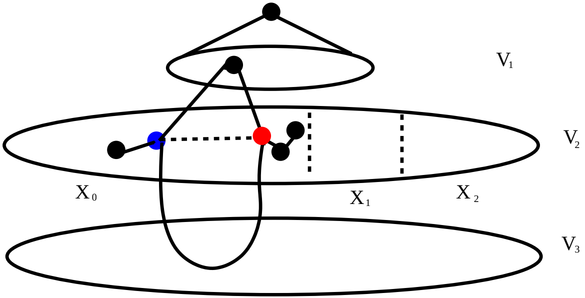


More Hitting Weak Spots



Trees in X_0 are connected via a path of length at most three through V_3.

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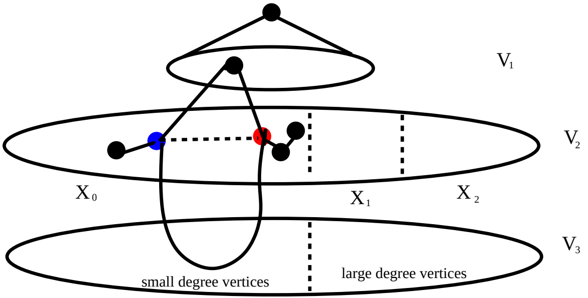

Small degree vertices can only "serve" so many trees of X_0. So, sum of large degree vertices is large.

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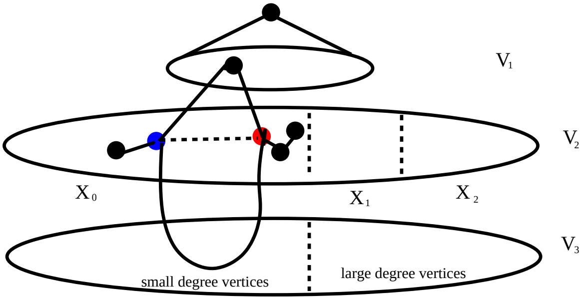

Small degree vertices can only "serve" so many trees of X_0. So, sum of large degree vertices is large. This allows us to add $\#(\text{trees in } X_0) - O(n^{3/4})$ edges to the count. Completes proof. $\square$

Open problems

Problem

Determine an exact result for $sat(n, K_{2,3})$.

Problem

Determine the asymptotic for $sat(n, K_{3,3})$.

Talk and results are available online at:
http://community.middlebury.edu/∼jschmitt/