Minimum Size of Bipartite-Saturated Graphs

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joint work with
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Definition
A graph $G$ is $F$-saturated if $F \not\subset G$,

$F \subset G + e$ for any $e \in E(\bar{G})$.

Problem
Determine the minimum number of edges, $\text{sat}(n, F)$, of an $F$-saturated graph.
Theorem (Erdős, Hajnal, Moon - 1964)

\[ sat(n, K_t) = (t - 2)(n - 1) - \binom{t - 2}{2} \]

Furthermore, the only $K_t$-saturated graph with this many edges is $K_{t-2} + \overline{K}_{n-t+2}$. 
Theorem (Ollmann - '72, Tuza - '86, Fisher, Fraughnaugh, Langley -'97)

\[ sat(n, C_4) = \left\lfloor \frac{3n - 5}{2} \right\rfloor, \quad n \geq 5 \]
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Theorem (Y.C. Chen, 07+)

\[
sat(n, C_5) = \left\lceil \frac{10n - 10}{7} \right\rceil, \quad n \geq 21
\]
Hamiltonian Cycles

Theorem

$$sat(n, C_n) = \left\lfloor \frac{3n + 1}{2} \right\rfloor, \quad n \geq 53$$

Bondy (’72) showed the lower bound. Clark, Entringer, Crane and Shapiro (’83-’86) gave upper bound based on Isaacs’ flower snarks (girth 5, 6). L. Stacho (’96) gave further constructions based on the Coxeter graph (girth 7).

Problem (Horák, Širáň -’86)

Is there a maximally non-hamiltonian graph of girth at least 8 (that meets this bound)?
- Conjecture (Bollobás - '78)

\[ n + c_1 \frac{n}{l} \leq \text{sat}(n, C_l) \leq n + c_2 \frac{n}{l} \]

- Theorem (Barefoot, Clark, Entringer, Porter, Székely, Tuza - '96)

\[ (1 + \frac{1}{2l + 8})n \leq \text{sat}(n, C_l) \]
Theorem (Barefoot et al. - ’96)

\[
sat(n, C_l) \leq (1 + \frac{6}{l-3})n + O(l^2) \text{ for } l \text{ odd, } l \geq 9
\]

\[
sat(n, C_l) \leq (1 + \frac{4}{l-2})n + O(l^3) \text{ for } l \text{ even, } l \geq 14
\]
Theorem (Barefoot et al. - ’96)

\[ \text{sat}(n, C_l) \leq \left(1 + \frac{1}{3} \frac{6}{l-3}\right)n + \frac{5l^2}{4} \quad \text{for } l \text{ odd, } l \geq 9, \ l \geq 17, \ n \geq 7l \]

\[ \text{sat}(n, C_l) \leq \left(1 + \frac{1}{2} \frac{4}{l-2}\right)n + \frac{5l^2}{4} \quad \text{for } l \text{ even } l \geq 14, \ l \geq 10, \ n \geq 3l \]
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Other values of $\text{sat}(n, F)$ known for:

- **matchings** (Mader - ’73),
- **paths and stars** (Kászonyi and Túza - ’86),
- **hamiltonian path,** $P_n$ (Frick and Singleton, 05; Dudek, Katona, Wojda - ’06)

$$\text{sat}(n, P_n) = \lceil \frac{3n - 2}{2} \rceil, \quad n \geq 54$$

- **longest path = detour** (Beineke, Dunbar, Frick, ’05)
Difficulties and Hereditary Properties Lacking

Quote from Erdős, Hajnal and Moon:
“One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult.”

- \( \text{sat}(n, F) \not\lesssim \text{sat}(n + 1, F) \)
- \( \mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow \text{sat}(n, \mathcal{F}_1) \geq \text{sat}(n, \mathcal{F}_2) \)
- \( F' \subset F \not\Rightarrow \text{sat}(n, F') \leq \text{sat}(n, F) \)
Best known upper bound

**Theorem (Kászonyi and Tuza)**

Let $F$ be a graph. Set

$$u := u(F) = |V(F)| - \alpha(F) - 1$$

$$s := s(F) = \min\{e(F') : F' \subseteq F, \alpha(F') = \alpha(F), |V(F')| = \alpha(F) + 1\}.$$

Then

$$\text{sat}(n, F) \leq (u + \frac{s - 1}{2})n - \frac{u(s + u)}{2}.$$  

They considered a clique on $u$ vertices joined to an $(s - 1)$-regular graph.
Problem

For an arbitrary graph \( F \), determine a non-trivial lower bound on \( \text{sat}(n, F) \).
Saturation for Bipartite Graphs, $K_{s,s}$
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Yields an improvement of a constant times $s^2$ over the KT bound.
Theorem (O. Pikhurko, S.)

There is a constant $C$ such that for all $n \geq 5$ we have

$$2n - Cn^{3/4} \leq \text{sat}(n, K_{2,3}) \leq 2n - 3.$$
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Proof of Lower Bound

Let $G$ be a $K_{2,3}$-saturated graph.
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Let $G$ be a $K_{2,3}$-saturated graph. If $\delta(G) \geq 4$, then $|E(G)| \geq 2n$ and we are done.
Proof of Lower Bound

Let $G$ be a $K_{2,3}$-saturated graph.

If $\delta(G) = 1$ then,
Proof of Lower Bound

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and so $|E(G)| \geq 2n - 3$. 
Otherwise, \( 2 \leq \delta(G) \leq 3 \), pick vertex of minimum degree and consider breadth-first search tree.
Proof of Lower Bound

Otherwise, $2 \leq \delta(G) \leq 3$, pick vertex of minimum degree and consider breadth-first search tree.

Tree has $n - 1$ edges, we must find $n - Cn^{3/4}$ more edges.
Divide and Conquer
Divide and Conquer
Hit 'em where they're weakest

$Y_0$ has at most one component which is a tree. Pick up an extra $V_3 - 1$ edges.
More Division and More Conquering

Pick up extra $V_2 - \#(\text{trees in } X_0)$ edges.
Introduction
Cycles
Paths, Matchings, Stars and General Bound
Complete Bipartite Graphs

More Division and More Conquering

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More Hitting Weak Spots

Trees in $X_0$ are connected via a path of length at most three through $V_3$. 
Small degree vertices can only “serve” so many trees of $X_0$. So, sum of large degree vertices is large.
More Hitting Weak Spots

Small degree vertices can only “serve” so many trees of $X_0$. So, sum of large degree vertices is large. This allows us to add

$\#(\text{trees in } X_0) - O(n^{3/4})$ edges to the count. Completes proof.
Open problems

**Problem**

*Determine an exact result for sat*(\(n, K_{2,3}\)).

**Problem**

*Determine the asymptotic for sat*(\(n, K_{3,3}\)).

Talk and results are available online at:
http://community.middlebury.edu/~jschmitt/