Graph pebbling in sparse graphs

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Graph pebbling is a technique developed for proving results in number theory, introduced by Lagarias and Saks and first used by Fan Chung.
Graph pebbling is a mathematical model for the transportation of consumable resources.
A pebbling move consists of removing two pebbles from a vertex and placing one of them on an adjacent vertex.
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Pebbling number of a graph $G$, denoted $\pi(G)$, is the least number of pebbles necessary to guarantee that, regardless of distribution of pebbles and regardless of target vertex, there exists a sequence of pebbling moves that enables us to place a pebble on the target vertex.
Lower Bound on the Pebbling Number

$$\pi(G) \geq \max\{ n, 2^{diam(G)} \}$$
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If $\pi(G) = n$, then $G$ is called *Class 0*.

Class 0 graphs include:

- the complete graph, $K_n$,
- the complete $t$-partite graph (except stars), $K_{p_1,p_2,\ldots,p_t}$,
- the $d$-dimensional hypercube, $Q_d$ (F. Chung, ’92),
- Petersen graph,
- Kneser graph (certain instances).
Problem:
Find necessary and sufficient conditions for $G$ to be Class 0.
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Most results have placed conditions on diameter and/or connectivity.
Class 0 graphs don’t have a cut-vertex
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Proposition (Blasiak, S.) If for any pair of non-adjacent vertices, \(u, \nu\), in \(G\) we have \(d(u) + d(\nu) \geq n\), then \(G\) is Class 0.

Corollary (Czygrinow, Hurlbert - '03) If \(\delta(G) \geq \lceil \frac{n}{2} \rceil\), then \(G\) is Class 0.
Our Result

Proposition (Blasiak, S.) If for any pair of non-adjacent vertices, \( u, v \), in \( G \) we have \( d(u) + d(v) \geq n \), then \( G \) is Class 0.

Corollary (Czygrinow, Hurlbert - '03) If \( \delta(G) \geq \lceil \frac{n}{2} \rceil \), then \( G \) is Class 0.

Theorem (Blasiak, S.) Let \( G \) be a graph on \( n \geq 6 \) vertices. If for each maximal independent set \( S \) of \( G \) we have

\[
\sum_{v \in S} d(v) \geq (|S| - 1)(n - |S|) + 2,
\]

then \( G \) is Class 0.
Showing sharpness

$|S|$, $n-|S|$,

independent set

cut-vertex

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Graph pebbling in sparse graphs
**Corollary** Complete multi-partite graphs (except stars) are Class 0.
Corollary (Pachter, Snevily, Voxman - ’95) Let $G$ be a connected graph with $n \geq 6$ vertices and $|E(G)|$ edges. If $|E(G)| \geq \left(\frac{n-1}{2}\right) + 2$, then $G$ is Class 0. And, the bound is sharp.
Edge density for Class 0

**Corollary** (Pachter, Snevily, Voxman - ’95) Let $G$ be a connected graph with $n \geq 6$ vertices and $|E(G)|$ edges. If $|E(G)| \geq \binom{n-1}{2} + 2$, then $G$ is Class 0. And, the bound is sharp.

**Proof:** As $\binom{n}{2} - (\binom{n-1}{2} + 2) = n - 3$, the hypothesis implies that $G$ has at most $n - 3$ non-edges.
Corollary (Pachter, Snevily, Voxman - '95) Let $G$ be a connected graph with $n \geq 6$ vertices and $|E(G)|$ edges. If $|E(G)| \geq \left(\frac{n-1}{2}\right) + 2$, then $G$ is Class 0. And, the bound is sharp.

Proof: As $\left(\frac{n}{2}\right) - \left(\frac{n-1}{2}\right) - 2 = n - 3$, the hypothesis implies that $G$ has at most $n - 3$ non-edges.
Corollary (Pachter, Snevily, Voxman - ’95) Let \( G \) be a connected graph with \( n \geq 6 \) vertices and \( |E(G)| \) edges. If \( |E(G)| \geq \binom{n-1}{2} + 2 \), then \( G \) is Class 0. And, the bound is sharp.

**Proof:** Thus,

\[
\sum_{v \in S} d(v) \geq |S|(n - |S|) - (n - 3 - \binom{|S|}{2})
\geq (|S| - 1)(n - |S|) + 2.
\]

Applying the theorem, \( G \) is Class 0.
What about the other extreme?

Determine the least number of edges in an $n$-vertex Class 0 graph, $f(n, 0)$.
Recall that this graph is Class 0, and so $f(n,0) \leq 2n - 4$ edges.
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Recall, we can’t have degree 1 vertices as Class 0 graphs don’t have cut-vertices.
How about minimum degree 2?

**Proposition**

In a graph $G$ on $n \geq 6$ (or 7) vertices, if vertex $v_1$ has $d(v_1) = 2$ (or 3), vertex $v_2$ has $d(v_2) = 2$ and $\text{dist}(v_1, v_2) \geq 3$ then $\pi(G) > n$.

Degree 2 vertices can’t be far apart.....

**Lemma**

Let $G$ be a Class 0 graph on $n \geq 6$ vertices, $X$ the set of degree two vertices of $G$, $Y := N[X] \setminus X$. One of the following conditions must hold:

1. $G[X] = P_3 = x_1x_2x_3$ and $x_1$ is adjacent to $y$, $x_3$ is adjacent to $y'$ with $y \neq y'$,

2. $G[X] = P_2 \cup (r - 2)P_1$ and $|Y| = 2$,

3. $G[X] = rP_1$ and there exists a vertex $y_1 \in Y$ which is adjacent to each vertex in $X$.

and so we can describe the structure of graphs that contain them.
Lemma

For $n \geq 10$, $f(n, 0) \geq \left\lfloor \frac{3n}{2} \right\rfloor$.

Proof: Applies previous lemma, first two cases are easy.
How about minimum degree 3? We don’t have an argument for how degree 3 vertices affect the structure of the graph.
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But surely there would be more edges in a Class 0 graph with minimum degree 3. Right?

Probably not.
So what’s the minimum number of edges in a graph with minimum degree 3 and fixed diameter?
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Bondy and Murty posed this question (recently).
Bondy and Murty knew that the minimum number of edges in a graph with minimum degree 3 and diameter two is:

And, we know this graph to be Class 0.
And, it was suggested to them by Erdős that for diameter 4 the following graph might be best:
A more interesting graph $\mathcal{P}$ with an equal number of edges is:

(a generalization of the Petersen graph).
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Degree Sum Condition
Cheap and No Class

Graph pebbling in sparse graphs
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John Schmitt
Graph pebbling in sparse graphs
Introduction
Class 0
Degree Sum Condition
Cheap and No Class
Graph pebbling in sparse graphs
Introduction
Class 0
Degree Sum Condition
Cheap and No Class

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**Proposition**

\[ \pi(\mathcal{P}) \leq n + 17. \]

*Proof:* Uses linear programming to find “witnesses”, and we take advantage of the symmetry of the graph.

**Conjecture**

\( \mathcal{P} \) *is Class 0.*

If \( \mathcal{P} \) is Class 0, then \( f(n, 0) \leq \frac{5(n-1)}{3} \).
Proposition
\[ \pi(\mathcal{P}) \leq n + 17. \]

Proof: Uses linear programming to find “witnesses”, and we take advantage of the symmetry of the graph.

Conjecture
\[ \mathcal{P} \text{ is Class 0.} \]

If \( \mathcal{P} \) is Class 0, then \( f(n, 0) \leq \frac{5(n-1)}{3} \).

Thanks!