Graph pebbling in sparse graphs

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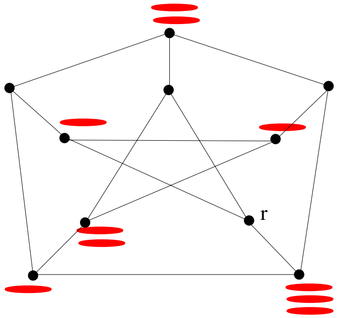
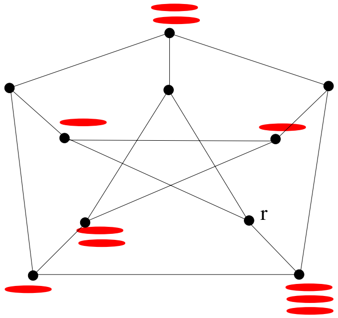
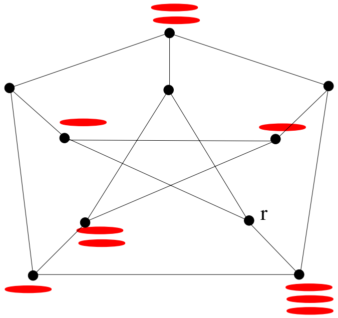
joint work with Anna Blasiak, Andrzej Czygrinow, David Herscovici, Glenn Hurlbert, Angelo Fu



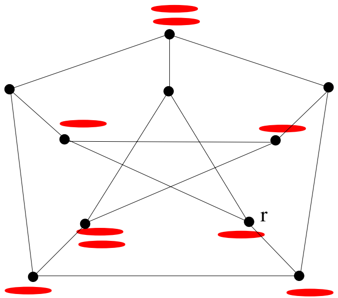
Graph pebbling is a technique developed for proving results in number theory, introduced by Lagarias and Saks and first used by Fan Chung.



Graph pebbling is a mathematical model for the transportation of consumable resources.



A pebbling move consists of removing two pebbles from a vertex and placing one of them on an adjacent vertex.



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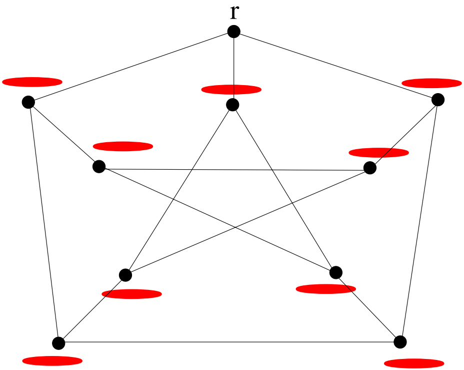
Pebbling number of a graph G, denoted $\pi(G)$, is the least number of pebbles necessary to guarantee that, regardless of distribution of pebbles and regardless of target vertex, there exists a sequence of pebbling moves that enables us to place a pebble on the target vertex.

Lower Bound on the Pebbling Number

$$\pi(G) \geq \max\{n, 2^{diam(G)}\}$$



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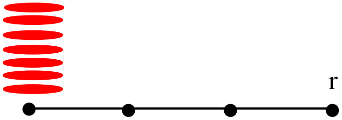

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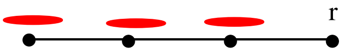

If $\pi(G) = n$, then G is called *Class 0*.

Class 0 graphs include:

- the complete graph, K_n,
- the complete t-partite graph (except stars), $K_{p_1,p_2,\ldots,p_t}$,
- the d-dimensional hypercube, Q_d (F. Chung, '92),
- Petersen graph,
- Kneser graph (certain instances).

Problem :
Find necessary and sufficient conditions for G to be Class 0.

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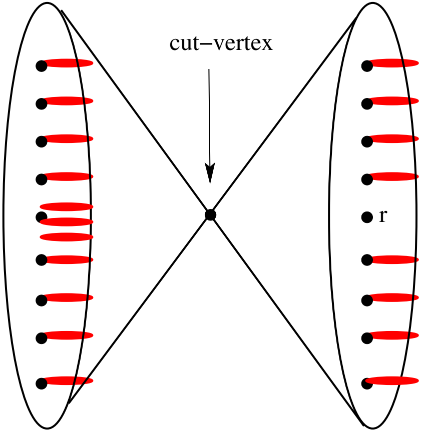
Most results have placed conditions on diameter and/or connectivity.

Class 0 graphs don't have a cut-vertex



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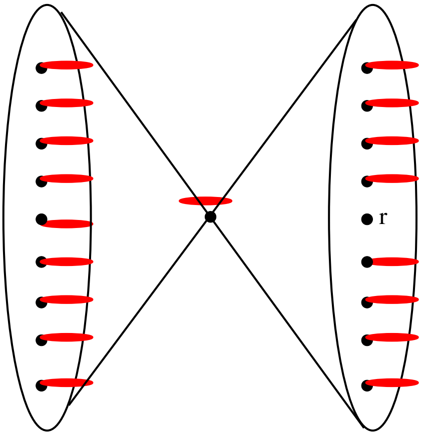

Our Result

Proposition (Blasiak, S.) If for any pair of non-adjacent vertices, u, v, in G we have $d(u) + d(v) \geq n$, then G is Class 0.

Corollary (Czygrinow, Hurlbert - '03) If $\delta(G) \geq \lceil \frac{n}{2} \rceil$, then G is Class 0.

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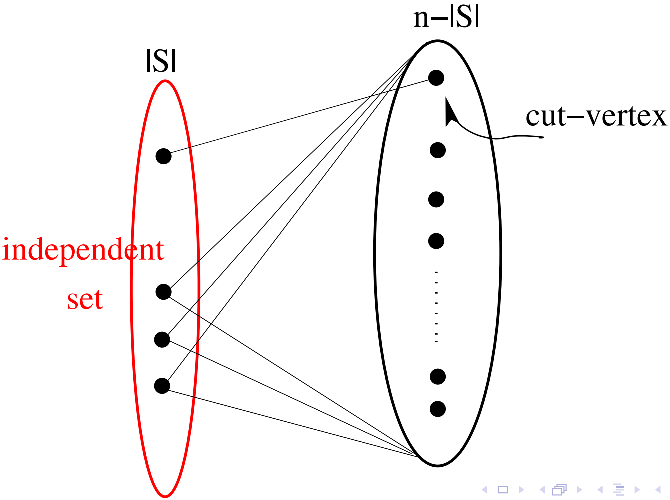
Theorem (Blasiak, S.) Let G be a graph on $n \geq 6$ vertices. If for each maximal independent set S of G we have

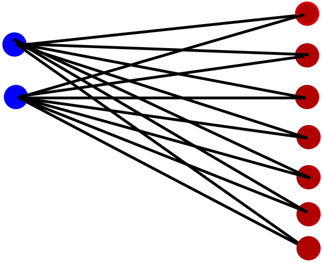
$$\sum_{v \in S} d(v) \quad \geq \quad (|S| - 1)(n - |S|) + 2,$$

then G is Class 0.

Showing sharpness



Corollary Complete multi-partite graphs (except stars) are Class 0.



Edge density for Class 0

Corollary (Pachter, Snevily, Voxman - '95) Let G be a connected graph with $n \geq 6$ vertices and $|E(G)|$ edges. If $|E(G)| \geq \binom{n-1}{2} + 2$, then G is Class 0. And, the bound is sharp.

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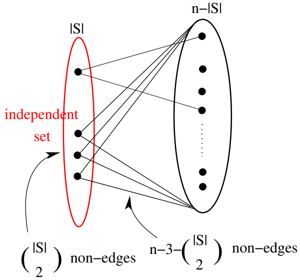

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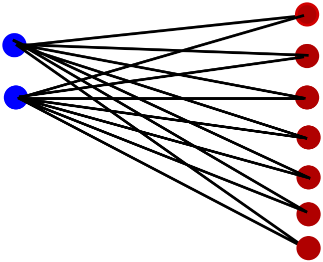
Proof: Thus,

$$\begin{aligned}\sum_{v \in S} d(v) &\geq |S|(n-|S|) - \left(n-3-\binom{|S|}{2}\right) \\ &\geq (|S|-1)(n-|S|)+2.\end{aligned}$$

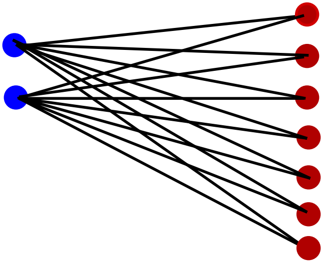
Applying the theorem, G is Class 0.

What about the other extreme?

Determine the least number of edges in an n-vertex Class 0 graph, $f(n, 0)$.



Recall that this graph is Class 0, and so $f(n, 0) \leq 2n - 4$ edges.



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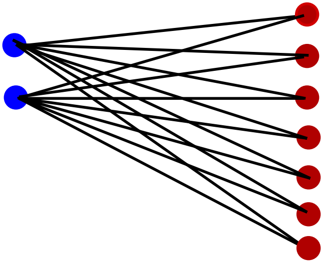

Recall that this graph is Class 0, and so $f(n,0) \leq 2n-4$ edges. So, the minimum degree of the graph we seek has to be less than 4.

Recall, we can't have degree 1 vertices as Class 0 graphs don't have cut-vertices.

How about minimum degree 2?

Proposition

In a graph G on $n \geq 6$ (or 7) vertices, if vertex v_1 has $d(v_1) = 2$ (or 3), vertex v_2 has $d(v_2) = 2$ and $dist(v_1, v_2) \geq 3$ then $\pi(G) > n$.

Degree 2 vertices can't be far apart.....

Lemma

Let G be a Class 0 graph on $n \geq 6$ vertices, X the set of degree two vertices of G, $Y := N[X] \setminus X$. One of the following conditions must hold:

1. $G[X] = P_3 = x_1x_2x_3$ *and* x_1 *is adjacent to* y, x_3 *is adjacent to* y' *with* $y \neq y'$,
2. $G[X] = P_2 \cup (r-2)P_1$ *and* $|Y| = 2$,
3. $G[X] = rP_1$ *and there exists a vertex* $y_1 \in Y$ *which is adjacent to each vertex in* X.

and so we can describe the structure of graphs that contain them.

Lemma

For $n \geq 10, f(n,0) \geq \lfloor \frac{3n}{2} \rfloor$.

Proof: Applies previous lemma, first two cases are easy.

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Probably not.

So what's the minimum number of edges in a graph with minimum degree 3 and fixed diameter?

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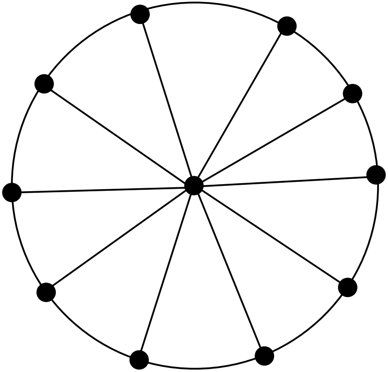
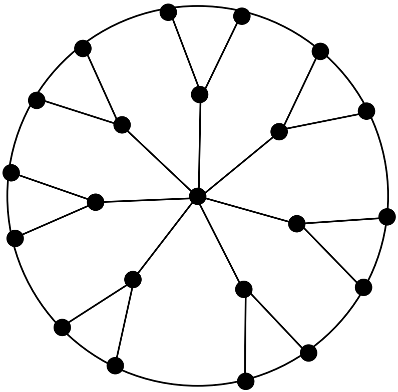
Bondy and Murty posed this question (recently).

Bondy and Murty knew that the minimum number of edges in a graph with minimum degree 3 and diameter two is:

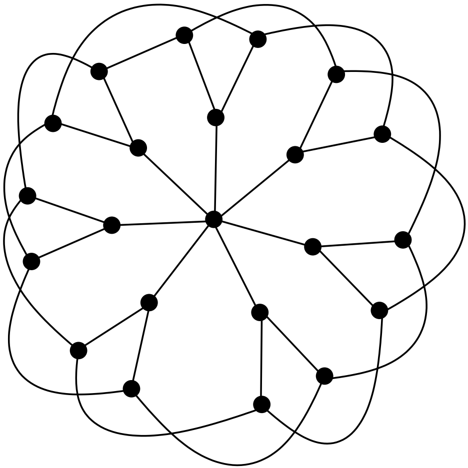


And, we know this graph to be Class 0.

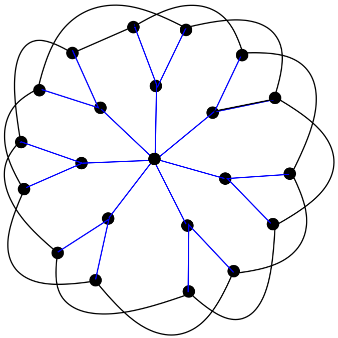
And, it was suggested to them by Erdős that for diameter 4 the following graph might be best:

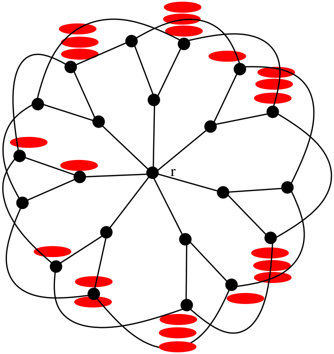


A more interesting graph $\mathcal{P}$ with an equal number of edges is:



(a generalization of the Petersen graph).

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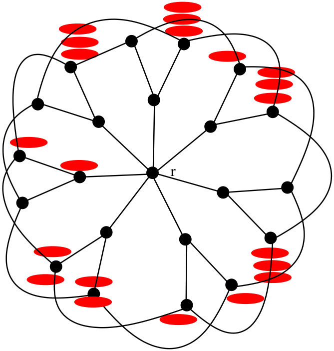
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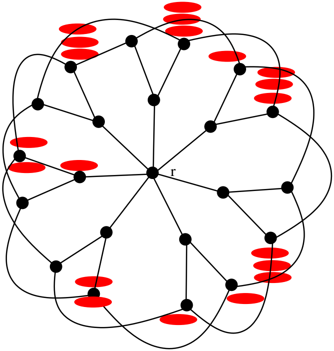
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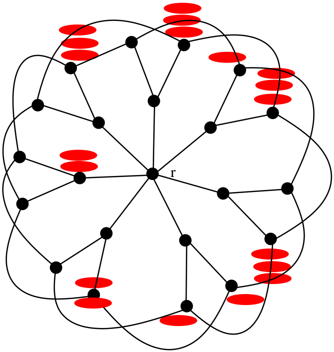
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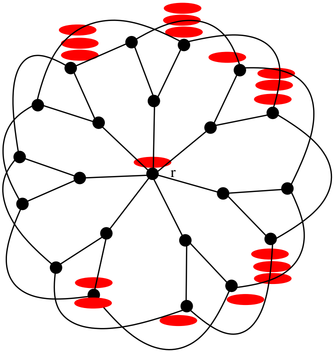
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Proposition

$\pi(\mathcal{P}) \leq n + 17.$

Proof: Uses linear programming to find "witnesses", and we take advantage of the symmetry of the graph.

Conjecture

$\mathcal{P}$ is Class 0.

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Thanks!