An extremal problem for a constant number of 1-factors

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joint work with
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Introduction
Definitions and notation
History
Overview

Results

Two 1-factors
Three 1-factors
$p \geq 4$ 1-factors
Threshold or not
Bounds

Questions

- $G$ — a graph of order $n$
- 1-factor — a spanning 1-regular subgraph (i.e. a perfect matching)
- $\Phi(G)$ — number of 1-factors in $G$
- $G_1 \cup G_2$ — the union of $G_1$ and $G_2$
- $G_1 + G_2$ — the join of $G_1$ and $G_2$
- $n$ — an even integer throughout

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\( n \))
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Theorem (G. Hetyei, unpublished - communicated by L. Lovász ('72))

The maximum number of edges in an $n$-vertex graph $G$ with a unique 1-factor (i.e. $\Phi(G) = 1$) is $\frac{n^2}{4}$. The $n$-vertex extremal graph $H_n$ is unique. For $n = 2$ it is $H_2 = K_2$ and for $n \geq 4$ we can define it recursively as $H_n = K_1 + (H_{n-2} \cup K_1)$. 
Theorem (G. Hetyei, unpublished - communicated by L. Lovász ('72))

The maximum number of edges in an n-vertex graph \( G \) with a unique 1-factor (i.e. \( \Phi(G) = 1 \)) is \( \frac{n^2}{4} \). The n-vertex extremal graph \( H_n \) is unique. For \( n = 2 \) it is \( H_2 = K_2 \) and for \( n \geq 4 \) we can define it recursively as \( H_n = K_1 + (H_{n-2} \cup K_1) \).

Proof.

There can be at most 2 edges joining two distinct edges of the 1-factor. Thus,

\[
|E(G)| \leq \frac{n}{2} + 2\left(\frac{n}{2}\right) = \frac{n^2}{4}.
\]
Various authors considered a generalization of Hetyei’s problem: what is the maximum number of edges in an $n$-vertex graph with unique $k$-factor? Results from:

- G. Hendry (’84)
- P. Johann (’00)
- L. Volkmann (’04)
- A. Hoffmann, L. Volkmann (’04)
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For extremal and structural results on 1-factors see:

- Bollobás’ *Extremal Graph Theory* (cf. Chapter 2)
- Lovász and Plummer’s *Matching Theory* (cf. Ch. 5 and 8)
The problem

**Problem**

*Determine maximum size and structure of such graphs when $\Phi(G) = p \geq 1$.***

**Notation:** Let $f(n, p)$ denote the maximum number of edges in $n$-vertex graph $G$ with $\Phi(G) = p$. 
The problem

**Problem**

*Determine maximum size and structure of such graphs when* \( \Phi(G) = p \geq 1. \)

**Notation:** Let \( f(n, p) \) denote the maximum number of edges in \( n \)-vertex graph \( G \) with \( \Phi(G) = p \).

e.g., \( f(n, 1) = \frac{n^2}{4} \)
Overview of our results

- Determine $f(n, p)$ for $2 \leq p \leq 6$ and find extremal graphs for $p = 2, 3$.
- Give bounds on $f(n, p)$ for arbitrary $p$, showing that $f(n, p)$ grows like $\frac{n^2}{4}$.
- Observe irregular and non-monotonic behavior of $f(n, p)$.
Basic fact

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Graph with \( p \) 1-factors
Lemma

Let $f(n, p) > 0$. Then $f(n + 2, p) \geq f(n, p) + (n + 1)$. Consequently, if $f(n, p) \geq \frac{n^2}{4} + c$ then $f(n + 2, p) \geq \frac{(n+2)^2}{4} + c$. 
Lemma

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Proof.

- Let $G_n$ be an extremal graph of order $n$ with $\Phi(G_n) = p$.
- Define recursively $G_{n+2} = K_1 + (G_n \cup K_1)$. Note that $\Phi(G_{n+2}) = \Phi(G_n) = p$. 
Lemma

Let \( f(n, p) > 0 \). Then \( f(n + 2, p) \geq f(n, p) + (n + 1) \).

Consequently, if \( f(n, p) \geq \frac{n^2}{4} + c \) then \( f(n + 2, p) \geq \frac{(n+2)^2}{4} + c \).

Proof.

- Let \( G_n \) be an extremal graph of order \( n \) with \( \Phi(G_n) = p \).
- Define recursively \( G_{n+2} = K_1 + (G_n \cup K_1) \). Note that \( \Phi(G_{n+2}) = \Phi(G_n) = p \).
- Hence, \( f(n + 2, p) \geq |E(G_{n+2})| = |E(G_n)| + (n + 1) = f(n, p) + (n + 1) \), as required.
$f(n, 2)$ - an extremal graph, $F_n$
Theorem

For every even $n \geq 4$, $f(n, 2) = \frac{n^2}{4} + 1$; otherwise $f(n, 2) = 0$. Furthermore, for every $n \geq 4$ there are precisely $\frac{n-2}{2}$ extremal graphs $G_n^i$, $1 \leq i \leq \frac{n-2}{2}$, defined recursively as follows:

$$G_n^i = \begin{cases} K_1 + (G_{n-2}^i \cup K_1) & \text{for } 1 \leq i \leq \frac{n-4}{2}, \\ F_n & \text{for } i = \frac{n-2}{2}. \end{cases}$$
Proof for $f(n, 2) = \frac{n^2}{4} + 1$

Agree on $k-1$ edges

Disagree elswhere
Proof for $f(n, 2) = \frac{n^2}{4} + 1$
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Proof for $f(n, 2) = \frac{n^2}{4} + 1$

Hetyei’s graph
Proof for \( f(n, 2) = \frac{n^2}{4} + 1 \)

Hetyei’s graph

No even chords.
Proof for $f(n, 2) = \frac{n^2}{4} + 1$

Hetyei’s graph

One odd chord per pair red edges.
Proof for $f(n, 2) = \frac{n^2}{4} + 1$
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\[
|E(G)| \leq (k - 1) + 2\binom{k - 1}{2} + (k - 1)(n - 2k + 2) + (n - 2k + 2)\left(\binom{n/2 - k + 1}{2}\right)
\]

\[
= -\frac{1}{2} \left( k - \frac{n - 1}{2} \right)^2 + \frac{n^2}{4} + \frac{9}{8} = g(k).
\]

Clearly, on the set \( \{0, \ldots, n/2 - 1\} \) the function \( g(k) \) is maximized when \( k = n/2 - 1 \). Thus
\[
|E(G)| \leq g(n/2 - 1) = n^2/4 + 1.
\]
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Analyze when equality holds to obtain extremal graphs. □
One can easily generalize the proof of above Theorem to get the following.

**Theorem**

*For every even $n \geq 4$, $f(n, 3) = \frac{n^2}{4} + 2$; otherwise $f(n, 3) = 0$. Furthermore, for each $n \geq 4$ there exists a unique extremal graph, for $n = 4$ it is $G_4 = K_4$ and for $n \geq 6$ it is given by $G_n = K_1 + (G_{n-2} \cup K_1)$.***
Lemma

Let $p \geq 1$ be an integer. Suppose that $f(n, r) \leq C$ for every $1 \leq r \leq p$. Then $f(n, p + 1) \leq C + 1$. 
Lemma

Let $p \geq 1$ be an integer. Suppose that $f(n, r) \leq C$ for every $1 \leq r \leq p$. Then $f(n, p + 1) \leq C + 1$.

Proof.

- Let $G$ be an $n$-vertex graph with $\Phi(G) = p + 1 \geq 2$ and $f(n, p + 1)$ edges. By way of contradiction we will assume that $f(n, p + 1) > C + 1$.
- We may find an edge $e$ in $G$ which belongs to at least one of the 1-factors but not to all of the 1-factors. Now consider $G - e$. 
Lemma

Let $p \geq 1$ be an integer. Suppose that $f(n, r) \leq C$ for every $1 \leq r \leq p$. Then $f(n, p + 1) \leq C + 1$.

Proof.

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- We may find an edge $e$ in $G$ which belongs to at least one of the 1-factors but not to all of the 1-factors. Now consider $G - e$.

- The graph $G - e$ contains $r$ 1-factor(s) for some $1 \leq r \leq p$ and has precisely $f(n, p + 1) - 1 > C \geq f(n, r)$ edges. This is a contradiction.
Hetyei’s Theorem and this Lemma immediately imply the following.

**Corollary**

For every $p \geq 1$, $f(n, p) \leq \frac{n^2}{4} + (p - 1)$. 
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**Corollary**

For every \( p \geq 1 \), \( f(n, p) \leq \frac{n^2}{4} + (p - 1) \).

<table>
<thead>
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<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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</thead>
<tbody>
<tr>
<td>( c_p )</td>
<td>0</td>
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</table>

**Table:** \( f(n, p) = \frac{n^2}{4} + c_p \) for every even \( n \geq n_p \).
Remark

It would be nice to prove previous Lemma under a weaker condition, namely, assuming $f(n, p) \leq C$ only. Unfortunately, the function $f(n, p)$ is not monotonic in $p$. One can check that $f(8, 14) = 20 < 21 = f(8, 12)$. Thus in order to proceed in the proof of Lemma we have to assume that $f(n, r) \leq C$ for all $1 \leq r \leq p$. 
Theorem

For every even \( n \geq 6 \), \( f(n, 4) = \frac{n^2}{4} + 2 \); otherwise \( f(n, 4) = 0 \).
Theorem

For every even $n \geq 6$, $f(n, 4) = \frac{n^2}{4} + 2$; otherwise $f(n, 4) = 0$.

Proof uses approach in previous lemma (finding an edge belonging to two or three 1-factors), connectivity results (to be found in Bollobás’ text), and yields no information on structure of extremal graphs.
A threshold graph is a graph that can be constructed from a one-vertex graph by repeated applications of the following two operations: (1) addition of a single isolated vertex to the graph, or (2) addition of a single dominating vertex to the graph.
A *threshold graph* is a graph that can be constructed from a one-vertex graph by repeated applications of the following two operations: (1) addition of a single isolated vertex to the graph, or (2) addition of a single dominating vertex to the graph.

**Fact**

*Extremal graphs for* $p = 0, 1, 2, 3$ *are threshold graphs (and also the ones we know for 4).*

**Fact**

*Recursive construction maintains threshold property.*
Theorem

If $\Phi(G)$ is a prime at least 5, then $G$ is not a threshold graph.
A lower bound

(p−1) 4–cycles sharing a common edge
has p 1–factors
A lower bound

(p−1) 4–cycles sharing a common edge
has p 1–factors

Then apply recursive construction.
Theorem

For every $p \geq 4$ and even $n \geq 2p$, 

$$\frac{n^2}{4} - (p - 2)(p - 1) \leq f(n, p) \leq \frac{n^2}{4} + (p - 2).$$
Theorem

For every $p \geq 4$ and even $n \geq 2p$, 

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<table>
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</thead>
<tbody>
<tr>
<td>$c_p$</td>
<td>0</td>
<td>1</td>
<td>2</td>
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Table: $f(n, p) = \frac{n^2}{4} + c_p$ for every even $n \geq n_p$. 
Question

\[ f(n, (2t - 1)!!) \geq \frac{n^2}{4} + (t^2 - t) \text{ and is tight for } t = 1 \text{ and } t = 2. \text{ Is it always?} \]

Question

We know \( f(n + 2, p) \geq f(n, p) + (n + 1) \). \text{Is it tight? If yes, then answer to previous question is yes.}
**Question**

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*We know* \( f(n + 2, p) \geq f(n, p) + (n + 1). \text{ Is it tight? If yes, then answer to previous question is yes.*

Thanks!