An extremal problem for a constant number of 1-factors

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joint work with
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- G — a graph of order n
- *1-factor* — a spanning 1-regular subgraph (i.e. a *perfect matching*)
- $\Phi(G)$ — number of 1-factors in G
- $G_1 \cup G_2$ — the *union* of G_1 and G_2
- $G_1 + G_2$ — the *join* of G_1 and G_2
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- n — an even integer throughout (except when we simply use it as a letter of a word, e.g. even)

Theorem (G. Hetyei, unpublished - communicated by L. Lovász ('72))

The maximum number of edges in an n-vertex graph G with a unique 1-factor (i.e. $\Phi(G) = 1$) is $\frac{n^2}{4}$. The n-vertex extremal graph H_n is unique. For $n = 2$ it is $H_2 = K_2$ and for $n \geq 4$ we can define it recursively as $H_n = K_1 + (H_{n-2} \cup K_1)$.

Theorem (G. Hetyei, unpublished - communicated by L. Lovász ('72))

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Proof.

There can be at most 2 edges joining two distinct edges of the 1-factor. Thus,

$$|E(G)| \leq n/2 + 2\binom{n/2}{2} = \frac{n^2}{4}.$$

$\square$

Various authors considered a generalization of Hetyei's problem: what is the maximum number of edges in an n-vertex graph with unique k-factor? Results from:

- G. Hendry ('84)
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For extremal and structural results on 1-factors see:

- Bollobás' *Extremal Graph Theory* (cf. Chapter 2)
- Lovász and Plummer's *Matching Theory* (cf. Ch. 5 and 8)

The problem

Problem

Determine maximum size and structure of such graphs when $\Phi(G) = p \geq 1$.

Notation: Let $f(n, p)$ denote the maximum number of edges in n-vertex graph G with $\Phi(G) = p$.

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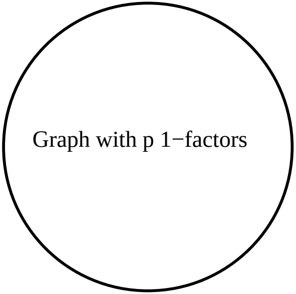
e.g., $f(n,1) = \frac{n^2}{4}$

Overview of our results

- Determine $f(n,p)$ for $2 \leq p \leq 6$ and find extremal graphs for $p = 2, 3$.
- Give bounds on $f(n,p)$ for arbitrary p, showing that $f(n,p)$ grows like $\frac{n^2}{4}$.
- Observe irregular and non-monotonic behavior of $f(n,p)$.

Basic fact



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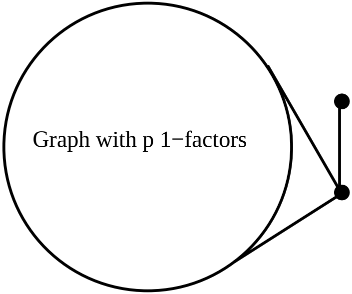

Lemma

Let $f(n,p) > 0$. Then $f(n+2,p) \geq f(n,p) + (n+1)$. Consequently, if $f(n,p) \geq \frac{n^2}{4} + c$ then $f(n+2,p) \geq \frac{(n+2)^2}{4} + c$.

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Proof.

- Let G_n be an extremal graph of order n with $\Phi(G_n) = p$.
- Define recursively $G_{n+2} = K_1 + (G_n \cup K_1)$. Note that $\Phi(G_{n+2}) = \Phi(G_n) = p$.

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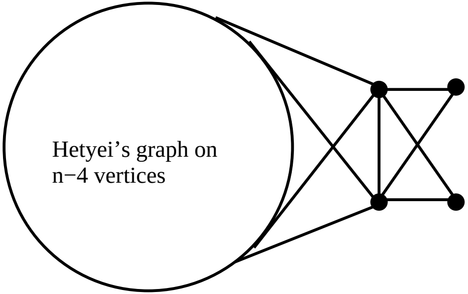
- Define recursively $G_{n+2} = K_1 + (G_n \cup K_1)$. Note that $\Phi(G_{n+2}) = \Phi(G_n) = p$.
- Hence, $f(n+2,p) \geq |E(G_{n+2})| = |E(G_n)| + (n+1) = f(n,p) + (n+1)$, as required.

$\square$

$f(n, 2)$ - an extremal graph, F_n

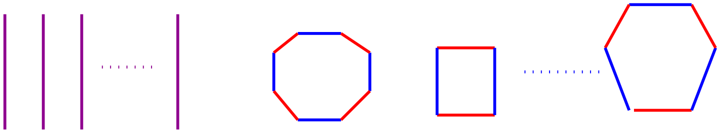


Theorem

For every even $n \geq 4$, $f(n,2) = \frac{n^2}{4} + 1$; *otherwise* $f(n,2) = 0$. *Furthermore, for every* $n \geq 4$ *there are precisely* $\frac{n-2}{2}$ *extremal graphs* G_n^i, $1 \leq i \leq \frac{n-2}{2}$, *defined recursively as follows:*

$$G_n^i = \begin{cases} K_1 + (G_{n-2}^i \cup K_1) & \text{for } 1 \leq i \leq \frac{n-4}{2}, \\ F_n & \text{for } i = \frac{n-2}{2}. \end{cases}$$

Proof for $f(n,2) = \frac{n^2}{4} + 1$



Agree on k−1 edges

Disagree elswhere

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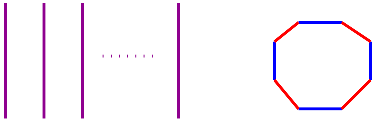

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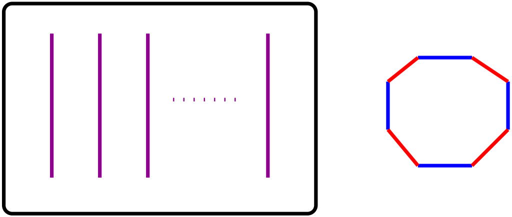

Hetyei's graph

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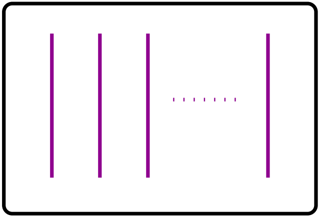

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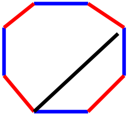

No even chords.

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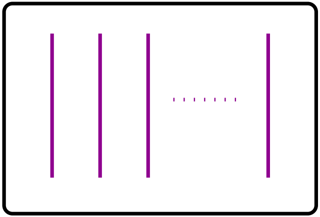

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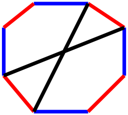

One odd chord per pair red edges.

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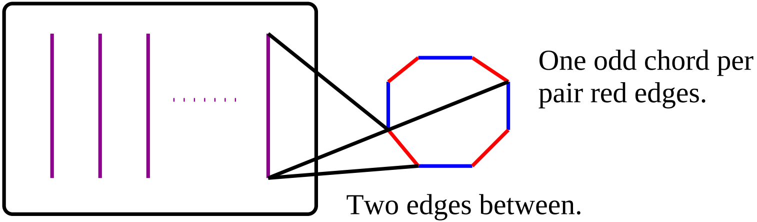

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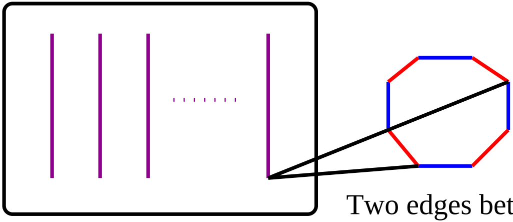

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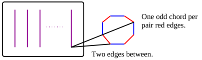

Hetyei's graph

$$
\begin{aligned}
|E(G)| &\leq \overbrace{(k-1)}^{r_1,\dots,r_{k-1}} + \overbrace{2\binom{k-1}{2}}^{r_i\ \ to\ \ r_j} + \overbrace{(k-1)(n-2k+2)}^{r_1,\dots,r_{k-1}\ \ to\ \ C} + \overbrace{(n-2k+2)}^{E(C)} \\
&\quad + \overbrace{\binom{n/2-k+1}{2}}^{Chords\ \ of\ \ C} \\
&= -\frac{1}{2}\left(k-\frac{n-1}{2}\right)^2 + \frac{n^2}{4} + \frac{9}{8} = g(k).
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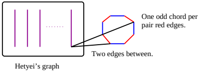

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&= -\frac{1}{2}\left(k - \frac{n-1}{2}\right)^2 + \frac{n^2}{4} + \frac{9}{8} = g(k).
\end{aligned}
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Clearly, on the set $\{0, \ldots, \frac{n}{2} - 1\}$ the function $g(k)$ is maximized when $k = n/2 - 1$. Thus $|E(G)| \leq g(n/2-1) = n^2/4 + 1$.

Analyze when equality holds to obtain extremal graphs. □

One can easily generalize the proof of above Theorem to get the following.

Theorem

For every even $n \geq 4$, $f(n,3) = \frac{n^2}{4} + 2$; otherwise $f(n,3) = 0$. Furthermore, for each $n \geq 4$ there exists a unique extremal graph, for $n = 4$ it is $G_4 = K_4$ and for $n \geq 6$ it is given by $G_n = K_1 + (G_{n-2} \cup K_1)$.

Lemma

Let $p \geq 1$ be an integer. Suppose that $f(n, r) \leq C$ for every $1 \leq r \leq p$. Then $f(n, p+1) \leq C + 1$.

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Proof.

- Let G be an n-vertex graph with $\Phi(G) = p + 1 \geq 2$ and $f(n, p+1)$ edges. By way of contradiction we will assume that $f(n, p+1) > C + 1$.
- We may find an edge e in G which belongs to at least one of the 1-factors but *not to all* of the 1-factors. Now consider $G - e$.

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- We may find an edge e in G which belongs to at least one of the 1-factors but *not to all* of the 1-factors. Now consider $G - e$.
- The graph $G - e$ contains r 1-factor(s) for some $1 \leq r \leq p$ and has precisely $f(n, p+1) - 1 > C \geq f(n, r)$ edges. This is a contradiction.

□

Hetyei's Theorem and this Lemma immediately imply the following.

Corollary

For every $p \geq 1$, $f(n, p) \leq \frac{n^2}{4} + (p - 1)$.

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p	1	2	3	4	5	6
c_p	0	1	2			
n_p	2	4	4			

Table: $f(n,p) = \frac{n^2}{4} + c_p$ for every even $n \geq n_p$.

$f(n,p)$ is not monotonic in p

Remark

It would be nice to prove previous Lemma under a weaker condition, namely, assuming $f(n,p) \leq C$ only. Unfortunately, the function $f(n,p)$ is not monotonic in p. One can check that $f(8,14) = 20 < 21 = f(8,12)$. Thus in order to proceed in the proof of Lemma we have to assume that $f(n,r) \leq C$ for all $1 \leq r \leq p$.

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Proof uses approach in previous lemma (finding an edge belonging to two or three 1-factors), connectivity results (to be found in Bollobás' text), and yields no information on structure of extremal graphs.

A *threshold graph* is a graph that can be constructed from a one-vertex graph by repeated applications of the following two operations: (1) addition of a single isolated vertex to the graph, or (2) addition of a single dominating vertex to the graph.

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Fact

Extremal graphs for $p = 0, 1, 2, 3$ *are threshold graphs (and also the ones we know for* 4*).*

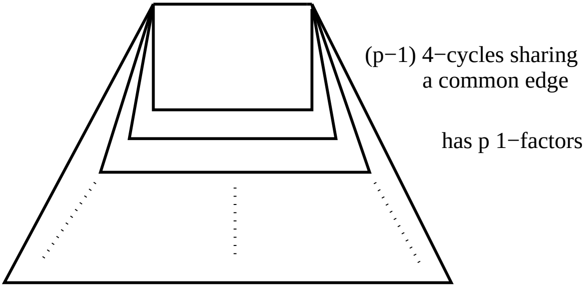
Fact

Recursive construction maintains threshold property.

Theorem

If $\Phi(G)$ is a prime at least 5, then G is not a threshold graph.

A lower bound



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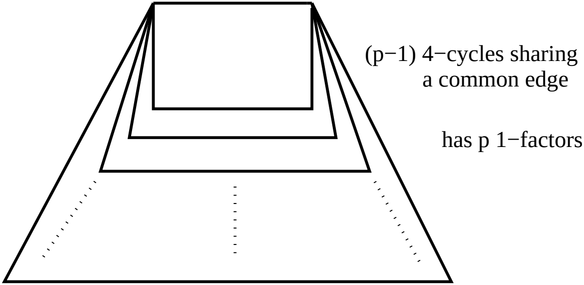

Then apply recursive construction.

Theorem

For every $p \geq 4$ and even $n \geq 2p$,

$$\frac{n^2}{4} - (p-2)(p-1) \leq f(n,p) \leq \frac{n^2}{4} + (p-2).$$

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p	1	2	3	4	5	6
c_p	0	1	2	2	2	3
n_p	2	4	4	6	6	6

Table: $f(n,p) = \frac{n^2}{4} + c_p$ for every even $n \geq n_p$.

Question

$f(n,(2t-1)!!) \geq \frac{n^2}{4} + (t^2 - t)$ and is tight for $t = 1$ and $t = 2$. Is it always?

Question

We know $f(n+2,p) \geq f(n,p) + (n+1)$. Is it tight? If yes, then answer to previous question is yes.

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Thanks!