

# Punctured and Structured Nullstellensätze

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# Nullstellensätze

# Hilbert's Nullstellensatz

**Theorem** (Hilbert 1893).

Let  $g_1, \dots, g_s, f \in \mathbb{C}[x_1, \dots, x_n]$ . Then  $f$  vanishes on all common zeros of  $g_1, \dots, g_s$  iff there are  $a_1, \dots, a_s \in \mathbb{C}[\mathbf{x}]$  and  $r \in \mathbb{N}$  such that  $f^r = a_1 g_1 + \dots + a_s g_s$ .

# Combinatorial Nullstellensätze

# Alon's Combinatorial Nullstellensatz I

## Theorem (Alon's Nullstellensatz I).

Let  $\mathbb{K}$  be a field,  $S = \times_{i=1}^n S_i$  with  $S_i \subseteq_{\text{fin}} \mathbb{K}$ . Then  $f \in \mathbb{K}[\mathbf{x}]$  vanishes on  $S$  iff there are  $a_1, \dots, a_s \in \mathbb{K}[\mathbf{x}]$  such that

$$f = a_1 g_1 + \dots + a_r g_r,$$

where  $g_i = \prod_{a \in S_i} (x_i - a)$  and  $\deg(a_i g_i) \leq \deg(f)$  for all  $i$ .

# Alon's Combinatorial Nullstellensatz II

**Theorem** (Alon's Combinatorial Nullstellensatz II).

Let  $\mathbb{K}$  be a field,  $S = \times_{i=1}^n S_i$  with  $S_i \subseteq_{\text{fin}} \mathbb{K}$ .

Let  $f \in \mathbb{K}[\mathbf{x}]$  be such that  $f$  contains a monomial  $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  with  $\alpha_i < |S_i|$  for all  $i$ .

If for all monomials  $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$  of  $f$  with  $\alpha \neq \gamma$  we have

$$\sum_{i=1}^n \gamma_i \leq \sum_{i=1}^n \alpha_i, \quad (\text{Alon's Condition})$$

then there is  $\mathbf{s} \in S$  with  $f(\mathbf{s}) \neq 0$ .

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**Applications:** Chevalley's Theorem, Cauchy Davenport Theorem, finding regular subgraphs, graph coloring,...

an “all-conquering force”... “unreasonably effective”

now with 441 citations on MathSciNet

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**Improvements:** Replace (Alon's Condition) with weaker conditions.



# Improved Combinatorial Nullstellensatz II

## Theorem (Combinatorial Nullstellensatz II).

Suppose that  $f \in \mathbb{K}[\mathbf{x}]$  contains a monomial  $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  with  $\alpha_i < |S_i|$  for all  $i$ .  
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then there is  $\mathbf{s} \in S$  with  $f(\mathbf{s}) \neq 0$ .

**Improvements:** Replace (Alon's Condition) with the following weaker conditions.

1. (Tao-Vu-Lason's Condition 2006)  $\exists i \in \underline{n} : \gamma_i \in [0, \alpha_i - 1]$ .
2. (Schauf's Condition 2008)  $\exists i \in \underline{n} : \gamma_i \in [0, \alpha_i - 1] \cup [\alpha_i + 1, |S_i| - 1]$ .

# Improved Combinatorial Nullstellensatz II

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**Applications from improvements:** Generalizations of earlier applications,  
Gordeev's 2024 results on Erdős box problem

# Structured Grids

# Structured Grids

**Definition** (Nica 2023).

$S \subseteq_{\text{fin}} \mathbb{K}$  is  **$\lambda$ -null**  $:\Leftrightarrow$  in  $\prod_{a \in S} (x - a)$ , the coefficients of  $x^{|S|-1}, \dots, x^{|S|-\lambda}$  are zero.

## Examples

- ▶ Every finite  $S$  is 0-null.
- ▶  $\{x \in \mathbb{C} \mid x^n = 1\}$  is  $n - 1$ -null.
- ▶  $\{0\}, \emptyset$  are  $\mu$ -null for all  $\mu \in \mathbb{N}$ .
- ▶  $S$  is 1-null if  $\sum_{a \in S} a = 0$ .

## Theorem (Nica 2023).

Let  $\mathbb{K}$  be a field,  $S = \times_{i=1}^n S_i$  such that  $S_i \subseteq_{\text{fin}} \mathbb{K}$  and  $S_i$  is  $\lambda_i$ -null.

Let  $f \in \mathbb{K}[\mathbf{x}]$  be such that  $f$  contains a monomial  $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  with  $\alpha_i < |S_i|$  for all  $i$ .

If for all monomials  $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$  of  $f$  with  $\alpha \neq \gamma$  we have

$$\sum_{i=1}^n \gamma_i \leq \min(\lambda_1, \dots, \lambda_n) + \sum_{i=1}^n \alpha_i, \quad (\text{Nica's Condition})$$

then there is  $\mathbf{s} \in S$  with  $f(\mathbf{s}) \neq 0$ .

**Applications from improvement:** Gardner's no-3-in-a-line problem (see Oh-S.-Wang 2025)

## Theorem (Nica 2023).

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then there is  $\mathbf{s} \in S$  with  $f(\mathbf{s}) \neq 0$ .

## Improvements:

### ► (Aichinger-S.-Zhan Condition)

$$\exists i \in \underline{n} : \gamma_i \in [0, \alpha_i - 1] \cup [\alpha_i + 1, \max(|S_i| - 1, \alpha_i + \lambda_i)].$$

# Comparison of the Nullstellensätze

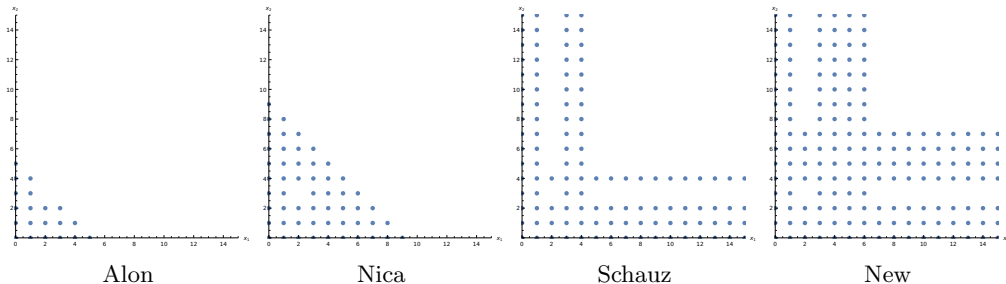
These theorems have in common:

- ▶ they guarantee a nonzero in a grid.
- ▶ the condition ensuring this is:
  1. there is a monomial  $\mathbf{x}^\alpha$  in  $f$  with  $\alpha_i < |S_i|$  for all  $i$ .
  2. all other monomials  $\mathbf{x}^\gamma$  of  $f$  are innocuous.

The more monomials one can declare innocuous, the better.

# Comparison of the Nullstellensätze

**Example.**  $S = \{(a, b) \in \mathbb{C}^2 \mid a^5 = b^5 = 1\}$ ,  $\lambda_1 = \lambda_2 = 4$ . Suppose  $f$  contains the monomial  $x_1^2 x_2^3$ . Then the following monomials are declared innocuous:



**Figure:**  $x_1^2 x_2^3 +$  any linear combination of the dotted monomials does not vanish on  $S = \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1^5 = x_2^5 = 1\}$ .



# Improved Nullstellensätze

## Generalizations and Improvements:

- ▶ **Multiplicity:**  $\mathbf{c}$  is a  $t$ -fold zero of  $f$  if all monomials of  $f_{\mathbf{c}} := f(c_1 + x_1, \dots, c_n + x_n)$  have total degree at least  $t$ . Ball and Serra (2009) provide theorems with bottom line:  
“Then there is  $\mathbf{s} \in S$  such that  $\mathbf{s}$  is not a  $t$ -fold zero of  $f$ .”
- ▶ **Multisets** (Kós and Rónyai 2012).
- ▶ **Beyond grids: Punctured Grids**  $X \setminus Y$ , where  $X, Y$  are grids. (Ball and Serra 2009)
- ▶ **Structured grids:** Use the property that an edge of the grid is  $\lambda$ -null. (Nica 2023)

Our recent manuscript provides combinations of these, for example a

**Structured Nullstellensatz for punctured grids.**

Manuscript: E.Aichinger, J.R.Schmitt, H.Zhan, *Structured and punctured Nullstellensätze*, arxiv 2025 (accepted to EJC).

# Structured Nullstellensätze for punctured grids

**Theorem** (A structured Nullstellensatz for punctured grids),  
Aichinger-S.-Zhan 2025.

Let  $X = \times_{i=1}^n X_i, Y = \times_{i=1}^n Y_i$  be grids over  $\mathbb{K}$  with  $Y_i \subseteq X_i$  and  $X_i, Y_i$   $\lambda$ -null for all  $i$ . Let

$$P := X \setminus Y.$$

Let  $f \in \mathbb{K}[x_1, \dots, x_n]$  with a monomial  $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  such that

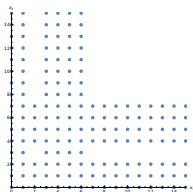
1. for all  $i$  :  $\alpha_i < |X_i|$ ,
2. there exists  $i$  such that  $\alpha_i < |X_i| - |Y_i|$ ,
3.  $\sum_{i=1}^n \alpha_i \geq \deg(f) - \lambda$ .

Then there is  $\mathbf{z} \in P$  with  $f(\mathbf{z}) \neq 0$ .

# Proofs

# Proof Ideas

- ▶ Given  $S \subseteq_{\text{fin}} \mathbb{K}^n$ , find generators  $G$  of the ideal  $\mathbb{I}(S) = \{f \in \mathbb{K}[\mathbf{x}] \mid f(\mathbf{a}) = 0 \text{ for all } \mathbf{a} \in S\}$ .
- ▶ We want to show that  $f \notin \mathbb{I}(S)$ .
- ▶ Show that  $\mathbf{x}^\alpha$  cannot disappear during **multivariate polynomial division** by  $G$  because
  - ▶  $\mathbf{x}^\alpha$  cannot be reduced by  $G$ .
  - ▶ All other monomials  $\mathbf{x}^\gamma$  cannot produce  $\mathbf{x}^\alpha$  **in the course of the division** – not in the first and not in any further step.
- ▶  $f$  has nonzero remainder by  $G$ : Then  $f \notin \langle G \rangle$  if  $G$  is a **Gröbner basis**.
- ▶ Use the  $S$ -Polynomial Theorem (Buchberger 1965) to show that  $G$  is indeed a Gröbner basis.



# Lower bounds for the number of nonzeros

# Alon-Füredi Nonzero Counting Theorem for punctured grids

- ▶ The Alon-Füredi Theorem gives a lower bound for the number of nonzeros of a polynomial on a grid.
- ▶ Alon-Füredi implies Warning's Second Theorem (Clark-Forrow-S. 2017).
- ▶ We have a version for punctured grids.

# Alon-Füredi Nonzero Counting Theorem for punctured grids

**Theorem** (Alon-Füredi for punctured grids), Aichinger-S.-Zhan 2025.

Let  $X = \times_{i=1}^n X_i$  and  $Y = \times_{i=1}^n Y_i$  be grids over the field  $\mathbb{K}$  with  $Y_i \subseteq X_i$  for all  $i$ ,  
 $P := X \setminus Y$ ,  $f \in \mathbb{K}[x_1, \dots, x_n] \setminus \{0\}$ .

Let  $a_i := |X_i|$ ,  $b_i := |Y_i|$  and

$$A := \{(y_1, \dots, y_n) \in \mathbb{N}^n \mid$$

$$\forall i \in \underline{n} : 1 \leq y_i \leq a_i, \exists i \in \underline{n} : y_i > b_i, \text{ and } \sum_{i=1}^n y_i \geq \sum_{i=1}^n a_i - \deg(f)\}.$$

If  $P \setminus \mathbb{V}(f) \neq \emptyset$ , then

$$|P \setminus V(f)| \geq \min\left\{\prod_{i=1}^n y_i - \prod_{i=1}^n \min(y_i, b_i) \mid (y_1, \dots, y_n) \in A\right\}.$$



Thank you!